Mathematics for Biology

MAT1142

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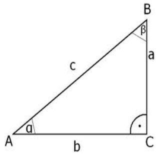
Trigonometry

Introduction

- **Trigonometry** is the study of the relations between the sides and angles of triangles.
- The word "trigonometry" is derived from the Greek words **trigono**, meaning "triangle", and **metro**, meaning "measure".



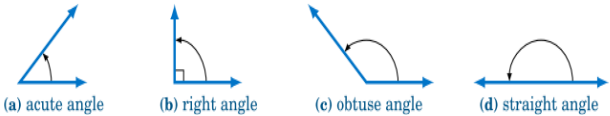
Angles

(a) An angle is **acute angle** if it is between 0°and 90°.

(b) An angle is a **right angle** if it equals 90°.

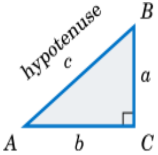
(c) An angle is **obtuse angle** if it is between 90°and 180°.

(d) An angle is a **straight angle** if it equals 180°.



Right triangle trigonometry

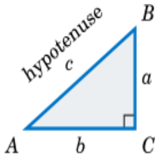
- If one of the angles is a right angle, then it is calld **right triangle**.
- In a right triangle, the side opposite the right angle is called the **hypotenuse**.
- The other two sides are called its **legs**.



Right triangle trigonometry

Cont...

- In Figure the right angle is C.
- The hypotenuse is the line segment $\overline{AB}$, which has length c.
- $\overline{BC}$ and $\overline{AC}$ are the legs, with lengths a and b, respectively.



Degrees versus radians

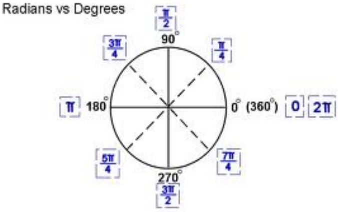
One revolution is 360°, and is also 2π radians. Thus, due to linear proportionality of the two scales, $1^\circ$to radians is:

$$\begin{aligned} 360^\circ &= 2\pi \text{ rad} \\ 1^\circ &= \frac{2\pi}{360^\circ} \text{ rad} \\ 1^\circ &= \frac{\pi}{180^\circ} \text{ rad} \end{aligned}$$

Degrees versus radians

Cont...

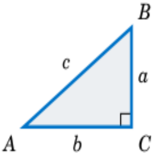
Degree	Radians
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π



Pythagorean Theorem

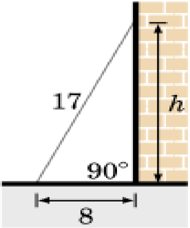
- The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.
- By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the Pythagorean Theorem.
- Thus, if a right triangle has a hypotenuse of length c and legs of lengths a and b, as in Figure, then the Pythagorean Theorem says:

$$\mathbf{a^2 + b^2 = c^2}.$$



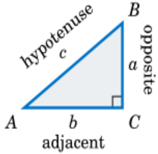
Example

A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. At what height is the top of the ladder touching the wall?



Trigonometric functions of an acute angle

- ABC is right triangle with the right angle at C and with lengths a, b, and c, as in the Figure.
- For the acute angle A, call the leg $\overline{BC}$ its **opposite side**, and call the leg AC its **adjacent side**.
- The **hypotenuse** of the triangle is the side $\overline{AB}$.
- We can define the six **trigonometric functions** of A as shown in next table.



Trigonometric functions of an acute angle

Table of trigonometric functions

Name of function	Abbreviation	Definition
sine A	$\sin A$	$= \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{a}{c}$
cosine A	$\cos A$	$= \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{b}{c}$
tangent A	$\tan A$	$= \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{a}{b}$
cosecant A	$\csc A$	$= \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{c}{a}$
secant A	$\sec A$	$= \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{c}{b}$
cotangent A	$\cot A$	$= \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{b}{a}$

Examples

1. A is an acute angle such that $\sin A = 3/5$. Find the values of the other trigonometric functions of A.

2. A is an acute angle such that $\tan A = 1/2$. Find the values of the other trigonometric functions of A.

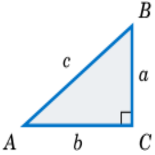
Basic trigonometric identities

- If equation is true for all angles θ for which both sides of the equation is defined, then it is called an **identity**.
- Trigonometric identities are identities involving the trigonometric functions.
- These identities are often used to simplify complicated expressions or equations.

Result 1

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} = \tan A$$

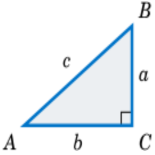
$$\tan A = \frac{\sin A}{\cos A}$$



Result 2

$$\frac{\cos A}{\sin A} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a} = \cot A$$

$$\cot A = \frac{\cos A}{\sin A}$$



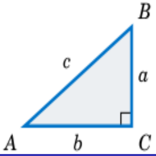
Result 3

$$\csc A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$

$$\sin A = \frac{1}{\csc A} \qquad \cos A = \frac{1}{\sec A} \qquad \tan A = \frac{1}{\cot A}$$

Result 4

$$\begin{aligned}
\sin A &= \frac{a}{c} \\
\cos A &= \frac{b}{c} \\
\sin^2 A + \cos^2 A &= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \\
&= \frac{a^2 + b^2}{c^2} \\
c^2 &= a^2 + b^2 \text{ (By Pythagorean Theorem)} \\
\sin^2 A + \cos^2 A &= \frac{c^2}{c^2} \\
\sin^2 A + \cos^2 A &= 1
\end{aligned}$$



Result 5

$$
\begin{aligned}
\sin^2 A + \cos^2 A &= 1 \\
\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} &= \frac{1}{\cos^2 A} \text{ (dividing by } \cos^2 A) \\
\tan^2 A + 1 &= \sec^2 A \\
\sec^2 A &= 1 + \tan^2 A
\end{aligned}
$$

Result 6

$$\begin{aligned}
\sin^2 A + \cos^2 A &= 1 \\
\frac{\sin^2 A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A} &= \frac{1}{\sin^2 A} \text{ (dividing by } \sin^2 A) \\
1 + \cot^2 A &= \csc^2 A \\
\csc^2 A &= 1 + \cot^2 A
\end{aligned}$$

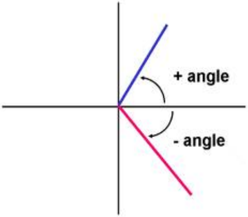
A table of exact trig values

θ	0 0°	$\pi/6$ 30°	$\pi/4$ 45°	$\pi/3$ 60°	$\pi/2$ 90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Negative angle identities

Identities also exist to relate the value of a trigonometric function at a given angle to the value of that function at the opposite of the given angle.

$$\begin{aligned}\sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta\end{aligned}$$



Negative angle identities

Examples

(i) $\sin(-30^\circ) = -\sin 30^\circ = -1/2$

(ii) $\cos(-45^\circ) = \cos 45^\circ = 1/\sqrt{2}$

(iii) $\tan(-60^\circ) = -\tan 60^\circ = -\sqrt{3}$

(iv) $\tan(-\pi/3) = -\tan \pi/3 = -\sqrt{3}$

(v) $\cos(-\pi/3) = \cos(\pi/3) = \dfrac{1}{2}$

Proof of identities

- In the proof of an identity we should strat from one side of the given identity and try to obtain the other side.
- It is usually best to start from more complicated side of the identity and atempt to simplify it.
- The sign $\equiv$ is somtimes used for identities.

Proof of identities

Examples

(i) $\sqrt{(1-\sin A)(1+\sin A)} = \cos A$

(ii) $\csc\theta\tan\theta = \sec\theta$

(iii) $\dfrac{1}{\sec^2\theta} + \dfrac{1}{\csc^2\theta} = 1$

(iv) $\cot\theta\sqrt{1-\cos^2\theta} = \cos\theta$

(v) $\sin\theta\tan\theta + \cos\theta = \sec\theta$

(vi) $\csc\theta - \sin\theta = \cot\theta\cos\theta$

(vii) $(\sin\theta+\cos\theta)^2 + (\sin\theta-\cos\theta)^2 = 2$

(viii) $(\sin\theta+\csc\theta)^2 = \sin^2\theta + \cot^3\theta + 3$

(ix) $\cos^4\theta - \sin^4\theta + 1 = 2\cos^2\theta$

(x) $\sec^4\theta - \sec^2\theta = \tan^4\theta + \tan^2\theta$

Proof of identities

Exercise

(i) $\cot s + 1 = \csc s\,(\cos s + \sin s)$

(ii) $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 + \sin x}{\cos x}$

(iii) $\sec x \sin x = \tan x$

(iv) $\sin x(\csc x - \sin x) = \cos^2 x$

(v) $\sin x \csc x \cos x = \cos x$

(vi) $\cot x \sin x \cos x = \cos^2 x$

(vii) $\csc x(\sin x + \tan x) = 1 + \sec x$

(viii) $1 - \dfrac{\sin x}{\csc x} = \cos^2 x$

(ix) $\dfrac{\cot x}{\csc x} = \cos x$

(x) $\sec x \csc x = \tan x + \cot x$

Addition formulas for sine and cosine

For the sum of any two angles A and B, we have the addition formulas:

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

Addition formulas for sine and cosine

Examples

(i) $\sin 75^\circ$

(iii) $\cos 75^\circ$

Subtraction formulas for sine and cosine

For the difference any two angles A and B, we have the subtraction formulas:

$$\begin{aligned}\sin(A-B) &= \sin A\cos B-\cos A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B\end{aligned}$$

Subtraction formulas for sine and cosine

Examples

(i) $\cos 15^\circ$

(ii) $\sin 15^\circ$

Addition formula for tangent

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\
&= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} \\
&= \frac{\tan A + \tan B}{1 - \tan A\tan B}
\end{aligned}$$

Subtraction formula for tangent

$$
\begin{aligned}
\tan(A-B) &= \frac{\sin(A-B)}{\cos(A-B)} \\
&= \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B} \\
&= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} + \dfrac{\sin A\sin B}{\cos A\cos B}} \\
&= \frac{\tan A - \tan B}{1 + \tan A\tan B}
\end{aligned}
$$

Addition and subtraction formulas for tangent

Example

(i) $\tan 75^\circ$

(ii) $\tan 15^\circ$

Example

Given angles A and B such that $\sin A = 4/5$, $\cos A = 3/5$, $\sin B = 12/13$ and $\cos B = 5/13$, find the exact values of followings:

(i) $\sin(A + B)$

(ii) $\cos(A - B)$

(iii) $\tan(A + B)$

Double-angle formulas

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

Double-angle formulas

Examples

1. Use a double angle identity to find the exact value of $\cos(2\pi/3)$.
2. Use a double angle identity to find the exact value of $\sin(2\pi/3)$.
3. Use a double angle identity to find the exact value of $\tan(2\pi/3)$.

Thank You