Mathematical Modelling-II

(AMT221β/IMT221β)

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Partial Differential Equations (PDEs)

The Concept of a PDE

Introduction

- Partial differential equations (PDEs) are used to describe a large variety of physical phenomena, from fluid flow to electromagnetic fields, aircraft simulation, and computer graphics.
- A PDE is any equation involving a function of more than one independent variable and at least one partial derivative of that function.
- **Eg:** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 1, \qquad \dfrac{\partial T}{\partial u} = 3w^2 \dfrac{\partial T}{\partial w} - 5v \dfrac{\partial T}{\partial v}.$

Definition
First order partial derivatives

The partial derivative of $f(x, y)$ with respect to x is

$$\frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

The partial derivative of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Definition
Second order partial derivatives

The second order partial derivatives of $f(x, y)$ are:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \lim_{h \to 0} \frac{f_x(x+h, y) - f_x(x, y)}{h}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \lim_{h \to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = \lim_{h \to 0} \frac{f_y(x, y+h) - f_y(x, y)}{h}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}$$

Examples

Some examples of PDEs are:

$$u_x + u_y = 0 \Leftarrow \text{Transport equation} \quad (1)$$

$$u_{xx} + u_{yy} = 0 \Leftarrow \text{Laplace's equation} \quad (2)$$

$$u_{tt} - u_{xx} = 0 \Leftarrow \text{Wave equation} \quad (3)$$

$$u_t - u_{xx} = 0 \Leftarrow \text{Heat equation} \quad (4)$$

$$u_{xx} + u_{yy} + u_{zz} = 0 \Leftarrow \text{Poisson equation} \quad (5)$$

$$u_t + uu_x + u_{xxx} = 0 \Leftarrow \text{KdV equation} \quad (6)$$

Classification of PDEs

- There are a number of properties by which PDEs can be separated into families of similar equations.
- The two main properties are the **order** and the **linearity.**

The order of a partial differential equation is the order of the highest derivative present in the equation. In examples above

- (1) is of first order,
- (2), (3), (4) and (5) are of second order,
- (6) is of third order.

A PDE is **linear** if it contains no products or powers of the unknown function or its partial derivatives. In our examples above

- (1), (2), (3), (4) are linear,
- (6) is nonlinear.

Classification of second order linear PDEs

A second order linear PDE in two variables has the general form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F. \tag{7}$$

The quantity $B^2 - AC$ is called the discriminant and it can be used to classify PDEs further as follows:

1. $B^2 - AC = 0 \Leftarrow$ The PDE is parabolic
2. $B^2 - AC < 0 \Leftarrow$ The PDE is elliptic
3. $B^2 - AC > 0 \Leftarrow$ The PDE is hyperbolic

Classification of second order linear PDEs
Example

Classify following PDEs based on the value of the discriminant:

(a) $u_{xx} + u_{yy} = 0$

(b) $u_t - u_{xx} = 0$

(c) $u_{xx} - u_{tt} = 0$

(a) $B^2 - AC = 0^2 - 1 \times 1 = -1 \Rightarrow$ Elliptic

(b) $B^2 - AC = 0^2 - (-1) \times 0 = 0 \Rightarrow$ Parabolic

(c) $B^2 - AC = 0^2 - (1) \times (-1) = 1 \Rightarrow$ Hyperbolic

Homogeneous PDEs

A PDE is **homogeneous** if every term involves the unknown function or its partial derivatives and **inhomogeneous** if it does not.

Eg:

- $u_t + cu_x = 0$ is homogeneous.
- The linear PDE is non-homogeneous

$$a(x,t)u_t + b(x,t)u_x + c(x,t)u = d(x,t),$$

unless $d(x,t) = 0$.

The superposition principle for PDEs

If two solutions, says u_1 and u_2 satisfy a linear homogeneous PDE, then any linear combination of them $u = c_1 u_1 + c_2 u_2$ is also a solution.

Example 1

Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}, \qquad c - \text{constant}. \tag{8}$$

Show that a solution is given by

$$w(t, x) = \cos(2x + 2ct).$$

Example 1
Solution

$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2}{\partial t^2}\cos(2x+2ct) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\cos(2x+2ct)\right) \\
&= \frac{\partial}{\partial t}\left(-\sin(2x+2ct)(2c)\right) \\
&= -2c\frac{\partial}{\partial t}\sin(2x+2ct) \\
&= -2c\cos(2x+2ct)\cdot 2c \\
&= -4c^2\cos(2x+2ct)
\end{aligned}
$$

Example 1
Solution ⇒ Cont...

$$
\begin{aligned}
w(t,x) &= \cos(2x+2ct) \\
\frac{\partial w}{\partial x} &= -2\sin(2x+2ct) \\
\frac{\partial^2 w}{\partial x^2} &= -4\cos(2x+2ct) \\
c^2\frac{\partial^2 w}{\partial x^2} &= -4c^2\cos(2x+2ct) \\
\text{L.H.S} &= \text{R.H.S} \\
\frac{\partial^2 w}{\partial t^2} &= c^2\frac{\partial^2 w}{\partial x^2}
\end{aligned}
$$

$w(t,x)$ is a solution of (8).

Example 2

Consider the Laplace equation given below:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Show that

(a) $\phi_1 = x$ and $\phi_2 = x^2 - y^2$ are solutions,

(b) a linear combination of ϕ_1 and ϕ_2 is also a solution,

of the Laplace equation.

(b)

$$
\begin{aligned}
\phi &= c_1\phi_1 + c_2\phi_2 \\
\phi &= c_1 x + c_2(x^2 - y^2) \\
\frac{\partial^2\phi}{\partial x^2} &= 2c_2 \\
\frac{\partial^2\phi}{\partial y^2} &= -2c_2 \\
\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0
\end{aligned}
$$

Initial Value Problems

- Generally partial differential equations have lots of solutions.
- To get a unique solution, we need some additional conditions.
- These conditions are usually in two varieties; **initial conditions** and **boundary conditions**.

- An initial condition specifies the physical state at a given time $t = t_0$.
- For example, an initial condition for the heat equation $u_t = ku_{xx}$, would be the starting temperature distribution:

$$u(x, 0) = f(x).$$

- PDEs are also generally only valid on a certain domain.
- Our heat equation was derived for a one-dimensional bar of length l, so the relevant domain in question can be taken to be the interval $0 < x < l$ and the boundary consists of the two points $x = 0$ and $x = l$.
- Boundary conditions specify how the solution is to behave on the boundary of this domain.

- We might know that, at the endpoints $x = 0$ and $x = l$, the temperatures $u(0, t)$ and $u(l, t)$ are fixed. Boundary conditions that give the value of the solution are called **Dirichlet conditions**.

- If the boundary conditions specify the derivative of the solution, they're called **Neumann conditions**. This would yield the boundary conditions $u_x(0, t) = u_x(l, t) = 0$ and meaning there should be no heat flow out of the boundary.

- We could also specify that we have one insulated end and at the other, we control the temperature; this is an example of a **mixed boundary condition**.

Analytical methods to solve PDEs

- There are variety of methods to obtain symbolic or closed form solutions of PDEs.
- The **method of separation of variables** is one such and it can be used to obtain analytical solutions for some simple PDEs.
- It should be noted that, this method cannot always be used.

Method of Separation of Variables

Outline of the method

1. **Separate the variables**

   Assume that

   $$u(x,t) = X(x)T(t).$$

   Substitute this into the PDE to get two ODE's for X and T separately.

2. **Decide on the sign of the separation constant**
   The constant arises when you separate the variables.

3. **Solve the separated ODE's**
   You get, for example, ODE's to solve for $X(x)$ and $T(t)$ that depend on the constant in Step 2.

Outline of the method
Cont...

4. **Solve the (homogeneous) boundary conditions**
   So that you know what $X(t)$ and $T(t)$ are, and reconstruct the funtion, for example $u(x,t)$ that you need, using $u(x,t) = X(x)T(t)$.

5. **Check** that the u(x, t) that you have actually solves the problem.

Example 1

Use separation of variables on the following partial differential equation:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, && (9)\\
u(0,t) &= 0, && (10)\\
u(l,t) &= 0, && (11)\\
u(x,0) &= f(x). && (12)
\end{aligned}
$$

We might suppose we have a separated solution, where

$$u(x,t) = X(x)T(t).$$

That is, our solution is the product of a function that depends only on x and a function that depends only on t.

Substituting this form into the PDE we get

$$\begin{aligned} \frac{\partial}{\partial t} X(x)T(t) &= k\frac{\partial^2}{\partial x^2} X(x)T(t) \\ X(x)T'(t) &= kX''(x)T(t) \end{aligned}$$

Now notice that we can move everything depending on x to one side and everything depending on t to the other.

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

On the left, we have an expression which depends only on t, while on the right, we have an expression that depends only on x.

Yet these two sides have to be equal for any choice of x and t we make. The only way this is possible is if both sides of the equation are the same constant. In other words,

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

We've written the minus sign explictly for convenience: it will turn out that $\lambda > 0$.

This constant λ is called the **separation constant**.

The equation above really contains a pair of separate ordinary differential equations:

$$
\begin{aligned}
X'' + \lambda X &= 0 \qquad (13)\\
T' + \lambda k T &= 0. \qquad (14)
\end{aligned}
$$

Now, in order that the product solution satisfy boundary condition. So we have

$$
\begin{aligned}
u(0,t) &= X(0)T(t) = 0 \Rightarrow X(0) = 0, \qquad (15)\\
u(l,t) &= X(l)T(t) = 0 \Rightarrow X(l) = 0. \qquad (16)
\end{aligned}
$$

We have got three cases to deal with.

Case I When $\lambda > 0$

Letting $\lambda = \beta^2$ for $\beta > 0$, the general solution of (13) is

$$X(x) = B\cos(\beta x) + C\sin(\beta x).$$

The general solution of (14) is

$$T(t) = Ae^{-\lambda kt}.$$

By (15), we obtained $X(0) = 0$, so

$$\begin{aligned} X(x) &= B\cos(\beta x) + C\sin(\beta x) \\ X(0) &= B\cos(\beta 0) + C\sin(\beta 0) \\ 0 &= B. \end{aligned}$$

By (16), we obtained $X(l) = 0$, so

$$\begin{aligned} X(x) &= B\cos(\beta x) + C\sin(\beta x) \\ X(l) &= 0\cos(\beta l) + C\sin(\beta l) \\ 0 &= C\sin(\beta l). \end{aligned}$$

To avoid only having the trivial solution, we must have $\beta l = n\pi$.

In other words,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \text{ for } n = 1, 2, 3, \cdots$$

So we end up having found an infinite number of solutions to our boundary value problem given by equations (9), (10) and (11), one for each positive integer n.

They are

$$u_n(x,t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right). \tag{17}$$

The heat equation is linear and homogeneous. As such, the principle of superposition still holds: a linear combination of solutions is again a solution. So any function of the form

$$u(x,t) = \sum_{n=0}^{N} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right). \tag{18}$$

is also a solution to (9), (10) and (11).

Notice that we haven't used our initial condition (12) yet, which is why we referred to (18) as a solutions to just the boundary value problem. How does our initial data come into play? We have

$$f(x) = u(x, 0) = \sum_{n=0}^{N} A_n \sin\left(\frac{n\pi x}{l}\right) \tag{19}$$

So if our initial condition has this form, (18) works perfectly for us, with the coeffcients A_n just being the associated coeffcients from $f(x)$.

Case II When $\lambda = 0$

The solution to the differential equation (13) is

$$X(x) = B + Cx.$$

Applying the boundary conditions gives,

$$\begin{aligned} 0 = X(0) &= B + C0 \Rightarrow B = 0 \\ 0 = X(l) &= 0 + Cl \Rightarrow C = 0. \end{aligned}$$

In this case the only solution is the trivial solution.

Case III When $\lambda < 0$

The solution to the differential equation is

$$X(x) = B\cosh\left(\sqrt{-\lambda}x\right) + C\sinh\left(\sqrt{-\lambda}x\right).$$

Applying boundary condtions gives

$$\begin{aligned} 0 &= X(0) = B \\ 0 &= X(l) = C\sinh(L\sqrt{-l}) \end{aligned}$$

Since $\lambda < 0$, so $L\sqrt{-l} \neq 0$.

Hence $C = 0$.

In this case the only solution is the trivial solution.

Example 2

Find the solution to the following heat equation problem on a rod of length 2.

$$
\begin{aligned}
u_t &= u_{xx} \\
u(0,t) &= u(2,t) = 0 \\
u(x,0) &= \sin\left(\frac{3\pi x}{2}\right) - 5\sin(3\pi x).
\end{aligned}
$$

Example 2
Solution

In this problem, we have $k = 1$ and $l = 2$. Now, we know that our solution will have the form of something like:

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{N} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right), \\
u(x,t) &= \sum_{n=0}^{N} A_n e^{-\left(\frac{n\pi}{2}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right). \\
t = 0 \Rightarrow u(x,0) &= \sum_{n=0}^{N} A_n \sin\left(\frac{n\pi x}{2}\right) \qquad (20)
\end{aligned}
$$

We just need to figure out which terms are represented and what the coefficients A_n are.

Example 2
Solution ⇒ Cont...

Our initial condition is

$$u(x,0) = \sin\left(\frac{3\pi x}{2}\right) - 5\sin(3\pi x). \qquad (21)$$

Looking at (20) and (21), we can see that the first term corresponds to $n = 3$ and the second $n = 6$, and there are no other terms.

Thus we have $A_3 = 1$, $A_6 = -5$, and all other $A_n = 0$. Our solution is then

$$u(x,t) = e^{-\left(\frac{9\pi^2}{4}\right)t}\sin\left(\frac{3\pi x}{2}\right) - 5e^{-(9\pi^2)t}\sin(3\pi x).$$

An infinite sum of separated solutions

- Let's consider what happens if we take an infinite sum of our separated solutions.

- Then our solution to boundary value problem is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right). \tag{22}$$

- Now the initial condition specifies that the coefficients must satisfy

$$f(x) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right). \tag{23}$$

- This idea is due to the French mathematician Joseph Fourier, and (23) is called the Fourier sine series for $f(x)$.

Example 3
Past paper 2011

Let a thin homogeneous string which is perfectly flexible under uniform tension lie its equilibrium position along the x-axis. The displacement $u(x,t)$ of vibrating string which is attached to the point $x = 0$ and $x = 2$ is described by the equation;

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad 0 \leq x \leq 2, \quad t > 0.$$

under the following conditions;

$$\begin{aligned} u(0,t) &= 0 = u(2,t) \\ u(x,0) &= \sin^3\left(\frac{\pi x}{2}\right) \\ u_t(x,0) &= 0 \end{aligned}$$

where c is the wave speed.

(i) By assuming $u(x,t) = X(x)T(t)$, in the usual notation build up two ordinary differential equations for $X(x)$ and $T(t)$. For non-trivial solutions of $u(x,t)$ obtain expressions for $X(x)$ and $T(t)$.

(ii) Show that the possible solution for the displacement of the string is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2}\cos\frac{n\pi ct}{2}$$

where A_n is a constant.

(iii) Determining A_n, obtain the solution for $u(x,t)$.

(i) Our intention is to try to find a solution that is a function of x times a function of t. That is, we write

$$u(x,t) = X(x)T(t).$$

Substituting this form into the PDE we get

$$X(x)T''(t) = c^2 X''(x)T(t)$$

which gives

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{24}$$

It is clear that the left-hand side of (24) is a function of time t, while the right hand side is a function of space x.

The only way that this can be true for all x and t is if both functions are actually equal to a constant. Hence

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = k \text{ (separation constant)} \quad (25)$$

$$X''(x) - kX(x) = 0 \quad (26)$$

$$T''(t) - c^2 kT(t) = 0 \quad (27)$$

The question remains what sign this constant should have.

Case I When $k > 0$

We take $k = \lambda^2$. Then

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \lambda^2 X = 0 \quad \Rightarrow \quad X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} - c^2\lambda^2 T = 0 \quad \Rightarrow \quad T(t) = c_3 e^{\lambda ct} + c_4 e^{-\lambda ct}$$

Therefore

$$u(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{\lambda ct} + c_4 e^{-\lambda ct})$$

Example 3

Solution ⇒ Cont...

Since $u(0, t) = 0$, we can get:

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u(0,t) &= X(0)T(t) \\ 0 &= X(0)T(t) \Rightarrow X(0) = 0 \end{aligned}$$

By considering the solution of ODE (26) with $X(0) = 0$, we get:

$$\begin{aligned} X(x) &= c_1 e^{\lambda x} + c_2 e^{-\lambda x} \\ 0 &= c_1 e^{\lambda 0} + c_2 e^{-\lambda 0} \\ 0 &= c_1 + c_2 \\ c_1 &= -c_2 \end{aligned}$$

And also since $u(2,t) = 0$, we can get:

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ u(2,t) &= X(2)T(t) \\ 0 &= X(2)T(t) \Rightarrow X(2) = 0 \end{aligned}$$

By considering the solution of ODE (26) with $X(2) = 0$, we get:

$$\begin{aligned} X(x) &= c_1 e^{\lambda x} + c_2 e^{-\lambda x} \\ X(2) &= c_1 e^{\lambda 2} + c_2 e^{-\lambda 2} \\ 0 &= c_1 (e^{2\lambda} - e^{-2\lambda}) \\ (e^{2\lambda} - e^{-2\lambda}) &\neq 0 \Rightarrow c_1 = c_2 = 0 \end{aligned}$$

This gives a trivial solution. Therefore $k > 0$ is not possible.

Case II When $k = 0$

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = 0 \Rightarrow X(x) = Ax + B$$

By considering the given boundary conditions, we have:

$$\begin{aligned} u(0,t) &= 0 \Rightarrow B = 0, \\ u(2,t) &= 0 \Rightarrow 2A = 0. \end{aligned}$$

Since we are looking for a non trivial solution $k = 0$ is not possible.

Example 3
Solution ⇒ Cont...

Case III When $k < 0$, say $k = -\lambda^2$.

The differential equations and solutions are

$$\begin{aligned}
\frac{d^2 X}{dx^2} + \lambda^2 X &= 0 \Rightarrow X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \\
\frac{d^2 T}{dt^2} + c^2 \lambda^2 T &= 0 \Rightarrow T(t) = c_3 \cos \lambda ct + c_4 \sin \lambda ct
\end{aligned}$$

Their general solution is

$$u(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) \cdot (c_3 \cos \lambda ct + c_4 \sin \lambda ct)$$

(ii) Using the condition $u(0,t) = 0$

We obtain $c_1 = 0$.

Then

$u_t(x,t) = c_2 \sin \lambda x \, (-c_3 \sin \lambda ct + c_4 \cos \lambda ct) \, \lambda c$

Since $u_t(x,0) = 0$

$$\begin{aligned} 0 &= c_2 \sin \lambda x (c_4) \lambda c \\ \Rightarrow c_4 &= 0 \end{aligned}$$

Therefore we have $u(x,t) = c_2 \sin \lambda x \cdot c_3 \cos \lambda ct$.

Since

$$\begin{aligned} u(2,t) = 0 \quad &\Rightarrow \quad \sin 2\lambda = 0 \\ &\Rightarrow \quad \lambda = \frac{n\pi}{2}; n = 1, 2, \cdots \end{aligned}$$

Thus, the possible solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} \cos \frac{n\pi ct}{2}.$$

(iii) Finally using condition $u(x,0) = \sin^3 \frac{\pi x}{2}$, we obtain

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} = \sin^3 \frac{\pi x}{2}$$
$$= \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$$

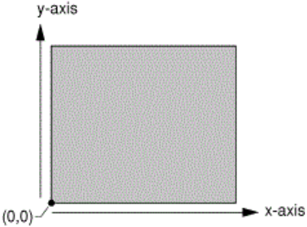
$A_1 = \frac{3}{4}$, $A_3 = -\frac{1}{4}$ while all other A_n's are zero.

Hence the required solution is

$$u(x,t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}.$$

Two dimensional heat flow

- Consider the heat flow in a metal plate of uniform thickness, in the directions parallel to length and breadth of the plate.
- There is no heat flow along the normal to the plane of the rectangle.



Let $u(x, y)$ be the temperature at any point (x, y) of the plate at time t is given by

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{28}$$

In the steady state, u does not change with t, so we have:

$$\frac{\partial u}{\partial t} = 0.$$

Hence (28) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is called **Laplace equation** in two dimensions.

How to solve Laplace equation

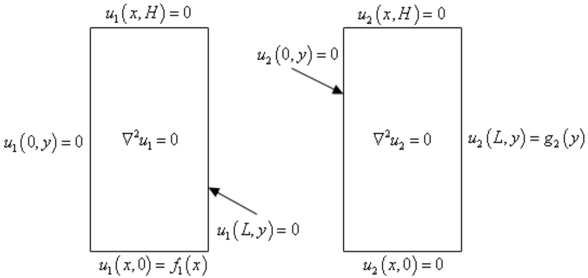
The way of solving Laplace's equation will depend upon the geometry of the 2-D object we're solving it on. Let's start out by solving it on the rectangle given by $0 \le x \le L$, $0 \le y \le H$. For this geometry Laplace's equation along with the four boundary conditions will be,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0, \qquad (29) \\ u(0,y) &= g_1(y) \quad u(L,y) = g_2(y), \\ u(x,0) &= f_1(x) \quad u(x,H) = f_2(x). \end{aligned}$$

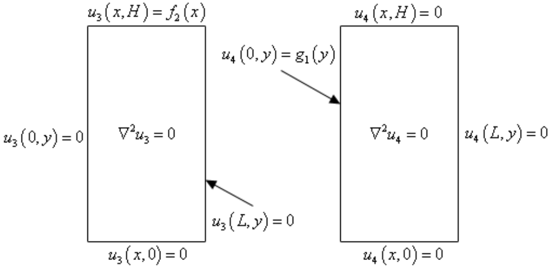
- It is important to notice that we will not have any initial conditions here.
- Both variables are spatial variables and each variable occurs in a 2^{nd} order derivative and so we'll need two boundary conditions for each variable.
- Moreover the partial differential equation is both linear and homogeneous; the boundary conditions are only linear and are not homogeneous.
- This creates a problem because separation of variables requires homogeneous boundary conditions.

- To completely solve Laplace's equation we're in fact going to have to solve it four times. Each time we solve it only one of the four boundary conditions can be non homogeneous while the remaining three will be homogeneous.
- The four problems are probably best shown with a quick sketch so let's consider the following sketch.

How to solve Laplace equation
Cont...



How to solve Laplace equation
Cont...



Now, once we solve all four of these problems the solution to our original system, (29), will be,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y).$$

Because we know that Laplace's equation is linear and homogeneous and each of the pieces is a solution to Laplace's equation then the sum will also be a solution. Also, this will satisfy each of the four original boundary conditions. We'll verify only the first one.

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0) = f_1(x).$$

How to solve Laplace equation
Cont...

- Here the nonhomogeneous boundary condition will take the place of the initial condition.
- We will apply separation of variables to the each problem and find a product solution that will satisfy the differential equation and the three homogeneous boundary conditions.
- Using the principle of superposition we'll find a solution to the problem and then apply the final boundary condition to determine the value of the constant(s) that are left in the problem.

Example

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which satisfies the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0$$

$$\text{and } u(x, a) = \sin \frac{n\pi x}{l}.$$

Example
Solution

The PDE is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{30}$$

Let

$$u(x, y) = X(x).Y(y). \tag{31}$$

Putting the value of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in (30) we have

$$\begin{aligned}
X''Y + XY'' &= 0 \\
\frac{X''}{X} &= -\frac{Y''}{Y} = -\lambda^2 \\
X'' &= -\lambda^2 X \quad \text{or} \quad X'' + \lambda^2 X = 0 && (32) \\
Y'' &= \lambda^2 Y \quad \text{or} \quad Y'' - \lambda^2 Y = 0 && (33)
\end{aligned}$$

By considering characteristic polynomial of (32), we get

$$\begin{aligned} m^2 + \lambda^2 &= 0 \\ m &= \pm i\lambda \\ X &= c_1 \cos \lambda x + c_2 \sin \lambda x. \end{aligned}$$

By considering characteristic polynomial of (33), we get

$$\begin{aligned} m^2 - \lambda^2 &= 0 \\ m &= \pm \lambda \\ Y &= c_3 e^{\lambda y} + c_4 e^{-\lambda y}. \end{aligned}$$

Putting the values of X and Y in (31) we have

$$u(x,y) = X(x).Y(y) \tag{34}$$
$$u(x,y) = (c_1 \cos\lambda x + c_2 \sin\lambda x).(c_3 e^{\lambda y} + c_4 e^{-\lambda y}). \tag{35}$$

Putting $x = 0$ and $u(0,y) = 0$ in (35) we have

$$\begin{aligned} u(0,y) &= (c_1 \cos\lambda 0 + c_2 \sin\lambda 0).(c_3 e^{\lambda y} + c_4 e^{-\lambda y}) \\ 0 &= c_1.(c_3 e^{\lambda y} + c_4 e^{-\lambda y}) \\ c_1 &= 0. \end{aligned}$$

Then (35) is reduced to

$$u(x,y) = c_2 \sin \lambda x (c_3 e^{\lambda y} + c_4 e^{-\lambda y}) \tag{36}$$

On putting $x = l$ and $u(l,y) = 0$, we have

$$\begin{aligned} u(l,y) &= c_2 \sin \lambda l (c_3 e^{\lambda y} + c_4 e^{-\lambda y}) \\ 0 &= c_2 \sin \lambda l (c_3 e^{\lambda y} + c_4 e^{-\lambda y}) \\ c_2 &\neq 0 \Rightarrow \sin \lambda l = 0 = \sin n\pi \\ \lambda l &= n\pi \\ \lambda &= \frac{n\pi}{l}. \end{aligned}$$

Now (36) becomes

$$u(x,y) = c_2 \sin\frac{n\pi x}{l}.\left(c_3 e^{\frac{n\pi y}{l}} + c_4 e^{\frac{-n\pi y}{l}}\right) \tag{37}$$

On putting $y = 0$ and $u(x,0) = 0$ in (37) we have

$$\begin{aligned} u(x,0) &= c_2 \sin\frac{n\pi x}{l}.\left(c_3 e^{\frac{n\pi 0}{l}} + c_4 e^{\frac{-n\pi 0}{l}}\right) \\ 0 &= c_2 \sin\frac{n\pi x}{l}.(c_3 + c_4) \\ (c_3 + c_4) &= 0 \quad \text{or} \quad c_3 = -c_4. \end{aligned}$$

(37) becomes

$$u(x,y) = c_2 c_3 \sin\frac{n\pi x}{l}.\left(e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}\right) \tag{38}$$

On putting $y = a$ and $u(x,a) = \sin\frac{n\pi x}{l}$ in (38), we get

$$\begin{aligned}
u(x,a) &= c_2 c_3 \sin\frac{n\pi x}{l}.\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right)\\
\sin\frac{n\pi x}{l} &= c_2 c_3 \sin\frac{n\pi x}{l}.\left(e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}\right)\\
c_2 c_3 &= \frac{1}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}}.
\end{aligned}$$

Putting this value in (38) we have

$$\begin{aligned}
u(x,y) &= \sin\frac{n\pi x}{l}\,\frac{e^{\frac{n\pi y}{l}} - e^{\frac{-n\pi y}{l}}}{e^{\frac{n\pi a}{l}} - e^{\frac{-n\pi a}{l}}}\\
u(x,y) &= \sin\frac{n\pi x}{l}\,\frac{\sinh\frac{n\pi y}{l}}{\sinh\frac{n\pi a}{l}}.
\end{aligned}$$

Exercise

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which satisfies the conditions

$$\begin{aligned} u(0, y) &= u(\pi, y) = 0 \text{ for all } y, \\ u(x, 0) &= k, \quad 0 < x < \pi. \end{aligned}$$

$$\text{and } \lim_{y \to \infty} u(x, y) = 0, \quad 0 < x < \pi.$$

Answer

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx e^{-ny}, \; k = \sum_{n=1}^{\infty} b_n \sin nx.$$

D'Alembert's Method

The wave equation

- The wave equation is second order linear hyperbolic PDE that describes the propagation of variety of waves, such as sound or water waves.
- Historically, the problem of a vibrating string such as that of a musical instrument was studied by Jean le Rond D'Alembert.
- He found a formula for general solution to the one-dimensional wave equation.
- It is named after him as **D'Alembert solutions**.

D'Alembert's solutions

- The wave equation $u_{tt} = c^2 u_{xx}$ has the D'Alembert's solutions $\phi(x - ct) + \psi(x + ct)$, for some choices of the functions ϕ and ψ to suit the given conditions.
- Such solutions are waves travelling at constant speed c on both directions along x-axis.
- To see how this is arrived; we need to do some applications of the chain rule for partial derivatives.

D'Alembert's solutions

Cont...

Let $v = x + ct$ and $w = x - ct$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial x} \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v}.(1) + \frac{\partial u}{\partial w}.(1) \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}
\end{aligned}$$

D'Alembert's solutions

Cont...

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial v} + \frac{\partial}{\partial w} \\
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)\left(\frac{\partial u}{\partial x}\right) \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right) \\
&= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \\
&= u_{vv} + 2u_{vw} + u_{ww}
\end{aligned}
$$

D'Alembert's solutions
Cont...

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial v}.\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}.\frac{\partial w}{\partial t} \\
\frac{\partial u}{\partial t} &= u_v c + u_w(-c) \\
u_t &= c(u_v - u_w) \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t}[c(u_v - u_w)] \\
&= c\left[c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\right](u_v - u_w) \\
&= c^2\left[\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial v \partial w}\right]
\end{aligned}
$$

D'Alembert's solutions

Cont...

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\
c^2 \left[\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2} - 2 \frac{\partial^2 u}{\partial v \partial w} \right] &= c^2 \left[\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial w^2} + 2 \frac{\partial^2 u}{\partial v \partial w} \right] \\
4c^2 \frac{\partial^2 u}{\partial v \partial w} &= 0 \\
\frac{\partial^2 u}{\partial v \partial w} &= 0, \quad c > 0 \\
u_{vw} &= 0. \qquad (39)
\end{aligned}
$$

By integrating (39) with respect to w we get

$$\frac{\partial u}{\partial v} = f(v) \tag{40}$$

where $f(v)$ is constant in respect of w.

Again integrating (40) with respect to v we get

$$u = \int f(v)\mathrm{d}v + \phi(w) \tag{41}$$

where $\phi(w)$ is a constant in respect of v.

$$\begin{aligned} u &= \psi(v) + \phi(w) \quad \text{where } \psi(v) = \int f(v)\mathrm{d}v \\ u(x,t) &= \psi(x+ct) + \phi(x-ct). \end{aligned} \tag{42}$$

This is D'Almbert's solution of wave equation.

To determine ϕ and ψ, let us apply initial conditions. Suppose we are given

$$\begin{aligned} u(x,0) &= f(x) \text{ and} \\ u_t(x,0) &= 0 \end{aligned}$$

Differentiating (42) with respect to t, we get

$$\begin{aligned} \frac{\partial u}{\partial t} &= -c\phi'(x-ct) + c\psi'(x+ct) \\ u_t(x,t) &= -c\phi'(x-ct) + c\psi'(x+ct) \\ u_t(x,0) &= -c\phi'(x-c0) + c\psi'(x+c0) \\ 0 &= -c\phi'(x) + c\psi'(x) \\ \phi'(x) &= \psi'(x) \end{aligned}$$

D'Alembert's solutions
Cont...

$$\begin{aligned}\phi'(x) &= \psi'(x)\\ \phi(x) &= \psi(x)+b, \qquad b-\text{constant}\end{aligned}$$

Again substituting $u(x,0)=f(x)$ and $t=0$ in (42) we get

$$\begin{aligned}u(x,t) &= \phi(x-ct)+\psi(x+ct)\\ u(x,0) &= \phi(x)+\psi(x)\\ f(x) &= \phi(x)+\psi(x)\\ f(x) &= [\psi(x)+b]+\psi(x)\\ f(x) &= 2\psi(x)+b\\ \psi(x) &= \frac{f(x)-b}{2} \qquad \phi(x)=\frac{f(x)+b}{2}\end{aligned}$$

On putting the values of $\psi(x+ct)$ and $\phi(x-ct)$ in (42), we get

$$\begin{aligned} u(x,t) &= \psi(x+ct)+\phi(x-ct) \\ &= \frac{f(x+ct)-b}{2}+\frac{f(x-ct)+b}{2} \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] \end{aligned}$$

Example 1

Use D'Alembert's method to find solutions for the PDE

$$\begin{aligned} 36y_{tt} &= 49y_{xx}, \text{ when} \\ y(x,0) &= \sin x \\ y_t(x,0) &= 2. \end{aligned}$$

Example 1
Solution

$$
\begin{aligned}
36y_{tt} &= 49y_{xx} \\
y_{tt} &= \frac{49}{36}y_{xx} \\
y_{tt} &= c^2 y_{xx} \Rightarrow \frac{49}{36}
\end{aligned}
$$

Example 1
Solution ⇒ Cont...

Let's take two new independent variable as $v = x + ct$ and $w = x - ct$.

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial v}.\frac{\partial v}{\partial x} + \frac{\partial y}{\partial w}.\frac{\partial w}{\partial x} \\
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial v}.(1) + \frac{\partial y}{\partial w}.(1) \\
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial v} + \frac{\partial y}{\partial w} \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial v} + \frac{\partial}{\partial w}
\end{aligned}$$

Example 1
Solution $\Rightarrow$ Cont...

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial v} + \frac{\partial}{\partial w} \\
\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) &= \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w}\right)\left(\frac{\partial y}{\partial x}\right) \\
\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial x}\right) + \frac{\partial}{\partial w}\left(\frac{\partial y}{\partial x}\right) \\
&= \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial v} + \frac{\partial y}{\partial w}\right) + \frac{\partial}{\partial w}\left(\frac{\partial y}{\partial v} + \frac{\partial y}{\partial w}\right) \\
&= \frac{\partial^2 y}{\partial v^2} + 2\frac{\partial^2 y}{\partial v \partial w} + \frac{\partial^2 y}{\partial w^2} \\
&= y_{vv} + y_{ww} + 2y_{vw}
\end{aligned}
$$

Example 1
Solution ⇒ Cont...

$$
\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial v}.\frac{\partial v}{\partial t} + \frac{\partial y}{\partial w}.\frac{\partial w}{\partial t} \\
\frac{\partial y}{\partial t} &= y_v c + y_w(-c) \\
y_t &= c(y_v - y_w) \\
\frac{\partial^2 y}{\partial t^2} &= \frac{\partial}{\partial t}[c(y_v - y_w)] \\
&= c\left[c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w}\right)\right](y_v - y_w) \\
&= c^2\left[\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial w^2} - 2\frac{\partial^2 y}{\partial v \partial w}\right]
\end{aligned}
$$

Example 1
Solution ⇒ Cont...

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2 \frac{\partial^2 y}{\partial x^2} \\
c^2\left[\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial w^2} - 2\frac{\partial^2 y}{\partial v \partial w}\right] &= c^2\left[\frac{\partial^2 y}{\partial v^2} + \frac{\partial^2 y}{\partial w^2} + 2\frac{\partial^2 y}{\partial v \partial w}\right] \\
4c^2 \frac{\partial^2 y}{\partial v \partial w} &= 0 \\
\frac{\partial^2 y}{\partial v \partial w} &= 0, \quad c > 0 \\
y_{vw} &= 0. \qquad (43)
\end{aligned}$$

Example 1
Solution ⇒ Cont...

By integrating (43) with respect to w we get

$$\frac{\partial y}{\partial v} = f(v) \tag{44}$$

where $f(v)$ is constant in respect of w.

Again integrating (44) with respect to v we get

$$y = \int f(v)\mathrm{d}v + \phi(w) \tag{45}$$

where $\phi(w)$ is a constant in respect of v.

$$\begin{aligned} y &= \phi(v) + \phi(w) \quad \text{where } \phi(v) = f(v)\mathrm{d}v \\ y(x,t) &= \phi(x+ct) + \psi(x-ct). \end{aligned}$$

This is D'Almbert's solution of wave equation.

To determine ϕ and ψ, let us apply initial conditions.

$$\begin{aligned} y(x,0) &= \sin x \text{ and} \\ y_t(x,0) &= 2. \end{aligned}$$

Differentiating with respect to t, we get

$$\begin{aligned} \frac{\partial y}{\partial t} &= -c\phi'(x-ct) + c\psi'(x+ct) \\ y_t(x,t) &= c\phi'(x+ct) - c\psi'(x-ct) \\ y_t(x,0) &= c\phi'(x+c0) - c\psi'(x-c0) \\ 2 &= c\phi'(x) - c\psi'(x) \\ \phi'(x) - \psi'(x) &= \frac{2}{c} \\ \phi(x) &= \psi(x) + \frac{2}{c}x + b \end{aligned}$$

Example 1

Solution ⇒ Cont...

Again substituting $y(x,0) = \sin x$ and $t = 0$ in () we get

$$
\begin{aligned}
y(x,t) &= \phi(x+ct) + \psi(x-ct) \\
y(x,0) &= \phi(x) + \psi(x) \\
\sin x &= \phi(x) + \psi(x) \\
\sin x &= \psi(x) + \psi(x) + \frac{2}{c}x + b \\
\sin x &= 2\psi(x) + \frac{2}{c}x + b \\
\psi(x) &= \frac{\sin x}{2} - \frac{x}{c} - \frac{b}{2} \\
\phi(x) &= \frac{\sin x}{2} + \frac{x}{c} + \frac{b}{2} \\
y(x,t) &= \frac{\sin(x+ct)}{2} + \frac{\sin(x-ct)}{2} + 2t
\end{aligned}
$$

Exercise 1

A string of length l is initially at rest in equilibrium position and each of its points are given velocity.

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}.$$

Find the displacement $y(x,t)$.

<u>Answer</u>

$$y(x,t) = \sum b_n \sin\frac{n\pi x}{l}\sin\frac{nc\pi}{l}t.$$

Method of characteristics

About method of characteristics

- The method of characteristics is a technique for solving first-order PDE.
- For a first-order PDE, the method of characteristics discovers curves (called **characteristic curves** or just **characteristics**) along which the PDE becomes an ODE.
- Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE.

Past paper 2011

Solve the initial value problem:

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= 0; \\ u(x,0) &= \sqrt{x}; \quad x > 0 \end{aligned}$$

using the method of characteristics.

$$u^2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0. \tag{46}$$

The aim of the method of characteristics is to solve the PDE by finding curves in the $x - t$ plane that reduce the equation into an ODE.

In general, any curve in the $x - t$ plane can be expressed in parametric form by $x = x(r)$, $t = t(r)$, where r gives a measure of the distance along the curve.

The curve starts at the initial point, $x = x_0$, $t = 0$, when $r = 0$.

$$u(x,t) = u(x(r), t(r)) \Rightarrow \ u \text{ is a function of } r.$$

Hence derivative of u with respect to r is

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}r} &= \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}r} + \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r} \\
\frac{\mathrm{d}u}{\mathrm{d}r} &= \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\partial u}{\partial x} + \frac{\mathrm{d}t}{\mathrm{d}r}\frac{\partial u}{\partial t} \qquad (47)
\end{aligned}$$

Compare (46) and (47)

$$\frac{\mathrm{d}u}{\mathrm{d}r} = 0 \qquad \frac{\mathrm{d}x}{\mathrm{d}r} = u^2 \qquad \frac{\mathrm{d}t}{\mathrm{d}r} = 1.$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 1 \Rightarrow t = r + k_1 \text{ (constant)}.$$

Since when $r = 0$, $t = 0 \Rightarrow k_1 = 0$.

Therefore $t = r$.

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}r} = 0 \quad \Rightarrow \quad u &= \text{constant on characteristics curve.} \\ u &= F(x_0)\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}r} &= u^2 \\
x &= u^2 r + k_2 \text{ (constant)}.
\end{aligned}
$$

When $r = 0$, $x = x_0$

$$
\begin{aligned}
x_0 &= k_2 \\
x &= u^2 t + x_0 \\
x_0 &= x - u^2 t
\end{aligned}
$$

Substituting for x_0 in $F(x_0)$, we obtain the implicit solution

$$u(x,t) = F(x - u^2 t),$$

where F is determined by given following initial conditions:

$$t = 0, \quad x = x_0 \quad \text{and} \quad u = \sqrt{x_0}. \tag{48}$$

Thus, $F(x_0) = \sqrt{x_0}$.

Therefore $u = \sqrt{x - u^2 t}$.

Squaring both sides,

$$\begin{aligned} u^2 &= x - u^2 t \\ x &= u^2(1+t) \\ u &= \sqrt{\frac{x}{1+t}},\ x > 0,\ t > 0. \end{aligned}$$

Thank you !