Mathematical Modelling-II

(AMT221β/IMT221β)

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Numerical Solutions of Systems of Differential Equations

Systems of Differential Equations

Why do we need systems of differential equations?

- Upto now, we only discussed individual first order differential equations.
- But it’s quite rare that a situation in the real world is modeled using only a single diffrential equation.
- The reason is, there are several interplaying factors at work in the evolution of something.

Why do we need systems of differential equations?
Example

- Suppose we are going to consider population dynamics.
- It would be possible to model the size of a single population using a single differential equation, making certain assumptions about death and birth rates (namely, that they are constant).
- But in general, this won't be the case: the death rate of a prey species is dependant on the size of a predator population and the size of the predator population will depend on the number of prey.

Why do we need systems of differential equations?

Example ⇒ Cont...

- To be able to write down a model for the size of the prey population, we need to know the predator population, and vice versa.
- This would then give us a system of two interlocked differential equations.

Here is an example of a system of first order, linear differential equations.

$$\frac{dy_1}{dt} = y_1 + 2y_2$$
$$\frac{dy_2}{dt} = 5y_1 + 5y_2$$

We call a system like this **coupled** because we need to know what y_1 is to know what y_2 is and vice versa.

Higher Order Differential Equations

Converting a higher order diffrential equation into a system

- Any higher order linear diffrential equation can be written as a system of first order differential equations.
- Let's see how this is done.

Example 1

Write the following 2^{nd} order differential equation as a system of first order, linear differential equations.

$$3y'' - 7y' + y = 0 \quad y(2) = 5 \quad y'(2) = -3.$$

We can convert second order differential equation into a system of first order differential equations by defining two new variables as follow:

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned}$$

If we differentiate both sides of these we get,

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = \frac{7}{3}y' - \frac{1}{3}y = \frac{7}{3}x_2 - \frac{1}{3}x_1. \end{aligned}$$

Example 1

Solution ⇒ Cont...

We can also convert the initial conditions over to the new variables.

$$\begin{aligned} x_1(2) &= y(2) = 5 \\ x_2(2) &= y'(2) = -3. \end{aligned}$$

Putting all of this together gives the following system of differential equations.

$$\begin{aligned} x_1' &= x_2 & \qquad x_1(2) = 5 \\ x_2' &= \frac{7}{3}x_2 - \frac{1}{3}x_1 & \qquad x_2(2) = -3. \end{aligned}$$

Example 2

Write the following 4^{th} order differential equation as a system of first order, linear differential equations.

$$y^{(4)} + ty''' - 5y'' - 2y' + y = 0.$$

We want to start by making an analogous change of variables as in above Example 1. The only difference is that, since our equation in this example is fourth order, we will need four new variables instead of just two.

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y''' \end{aligned}$$

If we differentiate both sides of these we get,

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = x_4 \\ x_4' &= y^{(4)} = -ty''' + 5y'' + 2y' - y = -tx_4 + 5x_3 + 2x_2 - x_1. \end{aligned}$$

Therefore, given 4^{th} order differential equation can be converted into a system of first order, linear differential equations as follows:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -tx_4 + 5x_3 + 2x_2 - x_1. \end{aligned}$$

Matrix differential equation

- A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders.
- A **matrix differential equation** contains more than one function stacked into vector form with a matrix relating the functions to their derivatives.

For example, a simple matrix ordinary differential equation is

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

where $\mathbf{x}(t)$ is $n \times 1$ vector of functions of an underlying variable t, $\mathbf{x}'(t)$ is the vector of first derivatives of these functions, and $\mathbf{A}$ is an $n \times n$ matrix, of which all elements are constants.

Example 1

Convert the following system to matrix from.

$$\begin{aligned} x_1' &= 5x_1 - 7x_2 \\ x_2' &= -2x_1 + 3x_2 \end{aligned}$$

Example 1
Solution

First we write the system as follow.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 5x_1 - 7x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$$

Now the right side can be written as a matrix multiplication,

$$\begin{aligned} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \mathbf{x}' &= \mathbf{A}\mathbf{x} \end{aligned}$$

Example 2

Convert the system into matrix form.

$$\begin{aligned} x_1' &= x_2 & x_1(2) = 5 \\ x_2' &= \frac{7}{3}x_2 - \frac{1}{3}x_1 & x_2(2) = -3. \end{aligned}$$

Example 2
Solution

The system is then,

$$\begin{aligned}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ \frac{7}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \mathbf{x}' &= \mathbf{A}\mathbf{x}\end{aligned}$$

The initial condition can also be written in matrix form as follows:

$$\mathbf{x}(2) = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

Euler's Method for System of Differential Equations

Euler's method

The Euler's method can be used to obtain approximations to the system of two first-order differential equations given below:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t, x, y), \quad t \in [a, b], \quad x(a) = x_0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t, x, y), \quad t \in [a, b], \quad y(a) = y_0 \end{aligned}$$

The Euler's method for the above system is

$$\begin{aligned} t_i &= a + ih \\ x_{i+1} &= x_i + hg(t_i, x_i, y_i) \text{ and} \\ y_{i+1} &= y_i + hf(t_i, x_i, y_i) \text{ for } i = 1, 2, \cdots, N-1. \end{aligned}$$

Example 1

Apply Euler's method to the first order system to compute approximations for $x(t)$ and $y(t)$ at time $t = 0.1$ and $t = 0.2$ with step size $h = 0.1$.

$$
\begin{aligned}
x' &= -2tx + 3y^2 \quad && x(0) = -1 \\
y' &= -3x^2(1 - y) \quad && y(0) = 2.
\end{aligned}
$$

Example 1
Solution

We are given that $t_0 = a = 0$, $x_0 = -1$, and $y_0 = 2$.

Using Euler's method we know that

$$\begin{aligned} t_i &= a + ih \\ x_{i+1} &= x_i + hg(t_i, x_i, y_i) \text{ and} \\ y_{i+1} &= y_i + hf(t_i, x_i, y_i) \text{ for } i = 1, 2, \cdots, N-1. \end{aligned}$$

Considering $i = 0$ we have

$$\begin{aligned} t_0 &= a = 0 \\ x_1 &= x_0 + hg(t_0, x_0, y_0) = -1 + 0.1(-2t_0x_0 + 3y_0^2) = 0.2 \\ y_1 &= y_0 + hf(t_0, x_0, y_0) = 2 + 0.1(-3x_0^2(1 - y_0)) = 2.3 \end{aligned}$$

Considering $i = 1$ we have

$$\begin{aligned} t_1 &= a + h = 0 + 0.1 = 0.1 \\ x_2 &= x_1 + hg(t_1, x_1, y_1) \approx 1.783 \\ y_2 &= y_1 + hf(t_1, x_1, y_1) \approx 2.3156 \end{aligned}$$

Example 2

Consider the second order differential equation

$$y'' + 2y' + y = e^{-t},$$

with initial conditions

$$y(0) = 1 \text{ and } y'(0) = 2.$$

Then by Euler's method with step size of $h = 0.25$, find $y(0.75)$.

Example 2

First, the second order differential equation has to be converted into two simultaneous first-order differential equations. Therefore, we take $y' = x$. Then

$$\begin{aligned} y'' + 2y' + y &= e^{-t} \\ x' + 2x + y &= e^{-t} \\ x' &= e^{-t} - 2x - y \end{aligned}$$

So the two simultaneous first order differential equations are

$$\begin{aligned} y' &= x & y(0) = 1 \\ x' &= e^{-t} - 2x - y & x(0) = 2. \end{aligned}$$

Example 2
Solution

Using Euler's method with $i = 0$, $t_0 = 0$, $y_0 = 1$, and $x_0 = 2$

$$\begin{aligned} t_0 &= a = 0 \\ x_1 &= x_0 + hg(t_0, x_0, y_0) = 1 \\ y_1 &= y_0 + hf(t_0, x_0, y_0) = 1.5 \end{aligned}$$

Using Euler's method with $i = 1$, $t_1 = 0.25$, $y_1 = 1.5$, $x_1 = 1$

$$\begin{aligned} t_1 &= 0.25 \\ x_2 &= x_1 + hg(t_1, x_1, y_1) = 0.31970 \\ y_2 &= y_1 + hf(t_1, x_1, y_1) = 1.75 \end{aligned}$$

Using Euler's method with $i = 2$, $t_2 = 0.5$, $y_2 = 1.75$, $x_2 = 0.31970$

$$\begin{aligned} t_2 &= 0.5 \\ x_3 &= x_2 + hg(t_2, x_2, y_2) = -0.1260 \\ y_3 &= y_2 + hf(t_2, x_2, y_2) = 1.8299 \end{aligned}$$

Modified Euler's Method for System of Differential Equations

Modified Euler's method

The Modified Euler's method can be used to obtain approximations to the system of two first-order differential equations given below:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= g(t,x,y), \quad t \in [a,b], \quad x(a) = x_0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t,x,y), \quad t \in [a,b], \quad y(a) = y_0\end{aligned}$$

The Modified Euler's method for the above system is

$$\begin{aligned}
k_{1,x} &= hg(t_i, x_i, y_i) \\
k_{1,y} &= hf(t_i, x_i, y_i) \\
k_{2,x} &= hg(t_{i+1}, x_i + k_{1,x}, y_i + k_{1,y}) \\
k_{2,y} &= hf(t_{i+1}, x_i + k_{1,x}, y_i + k_{1,y}) \\
x_{i+1} &= x_i + \frac{1}{2}(k_{1,x} + k_{2,x}) \\
y_{i+1} &= y_i + \frac{1}{2}(k_{1,y} + k_{2,y})
\end{aligned}$$

Exercise

Apply the Modified Euler method to the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y \qquad x(0) = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x \qquad y(0) = 1 \end{aligned}$$

with $h = 0.1$.

Fourth Order Runge-Kutta Method for System of Differential Equations

Fourth order Runge-Kutta method

The fourth order Runge-Kutta method can be used to obtain approximations to the system of two first-order differential equations given below:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= g(t, x, y), \quad t \in [a, b], \quad x(a) = x_0 \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t, x, y), \quad t \in [a, b], \quad y(a) = y_0 \end{aligned}$$

Fourth order Runge-Kutta method

Cont...

The fourth Order Runge-Kutta method for the above system is

$$
\begin{aligned}
k_{1,x} &= hg(t_i, x_i, y_i)\\
k_{1,y} &= hf(t_i, x_i, y_i)\\
k_{2,x} &= hg\left(t_i + \frac{h}{2}, x_i + \frac{k_{1,x}}{2}, y_i + \frac{k_{1,y}}{2}\right)\\
k_{2,y} &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_{1,x}}{2}, y_i + \frac{k_{1,y}}{2}\right)\\
k_{3,x} &= hg\left(t_i + \frac{h}{2}, x_i + \frac{k_{2,x}}{2}, y_i + \frac{k_{2,y}}{2}\right)\\
k_{3,y} &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_{2,x}}{2}, y_i + \frac{k_{2,y}}{2}\right)\\
k_{4,x} &= hg\left(t_i + h, x_i + k_{3,x}, y_i + k_{3,y}\right)\\
k_{4,y} &= hf\left(t_i + h, x_i + k_{3,x}, y_i + k_{3,y}\right)
\end{aligned}
$$

Fourth order Runge-Kutta method
Cont...

$$
\begin{aligned}
x_{i+1} &= x_i + \frac{1}{6}\left(k_{1,x} + 2k_{2,x} + 2k_{3,x} + k_{4,x}\right) \\
y_{i+1} &= y_i + \frac{1}{6}\left(k_{1,y} + 2k_{2,y} + 2k_{3,y} + k_{4,y}\right)
\end{aligned}
$$

Example

Let $i_1(t)$ and $i_2(t)$ are currents in a closed circuit at time t. Suppose the switch in the circuit is closed at time $t = 0$. Then $i_1(0) = 0$ and $i_2(0) = 0$. The system of equations for the circuit is as follows:

$$\begin{aligned} i_1' &= -4i_1 + 3i_2 + 6 \qquad i_1(0) = 0, \\ i_2' &= 0.6i_1' - 0.2i_2 \qquad i_2(0) = 0. \end{aligned}$$

Use the Runge-Kutta fourth order method with $h = 0.1$ to find $i_1(0.1)$ and $i_2(0.1)$.

The above system can be rewritten as:

$$
\begin{aligned}
i_1' = g(t, i_1, i_2) &= -4i_1 + 3i_2 + 6 \\
i_2' = f(t, i_1, i_2) &= 0.6i_1' - 0.2i_2 \\
&= 0.6(-4i_1 + 3i_2 + 6) - 0.2i_2 \\
&= -2.4i_1 + 1.6i_2 + 3.6
\end{aligned}
$$

with intial conditions

$$
\begin{aligned}
i_1(0) &= 0 \\
i_2(0) &= 0.
\end{aligned}
$$

Example
Solution ⇒ Cont...

We will apply the Runge-Kutta method of order four to this system with $h = 0.1$.

Since $x_0 = i_1(0) = 0$ and $y_0 = i_2(0) = 0$,

$$\begin{aligned}
k_{1,x} &= hg(t_0, x_0, y_0) \\
&= 0.1g(0, 0, 0) \\
&= 0.1[-4(0) + 3(0) + 6] = 0.6 \\
k_{1,y} &= hf(t_0, x_0, y_0) \\
&= 0.1f(0, 0, 0) \\
&= 0.1[-2.4(0) + 1.6(0) + 3.6] = 0.36
\end{aligned}$$

Example

Solution ⇒ Cont...

$$
\begin{aligned}
k_{2,x} &= hg\left(t_i + \frac{h}{2}, x_i + \frac{k_{1,x}}{2}, y_i + \frac{k_{1,y}}{2}\right) \\
&= hg\left(t_0 + \frac{h}{2}, x_0 + \frac{k_{1,x}}{2}, y_0 + \frac{k_{1,y}}{2}\right) \\
&= 0.1g(0.05, 0.3, 0.18) \\
&= 0.1[-4(0.3) + 3(0.18) + 6] = 0.534 \\
k_{2,y} &= hf\left(t_i + \frac{h}{2}, x_i + \frac{k_{1,x}}{2}, y_i + \frac{k_{1,y}}{2}\right) \\
&= hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_{1,x}}{2}, y_0 + \frac{k_{1,y}}{2}\right) \\
&= 0.1f(0.05, 0.3, 0.18) \\
&= 0.1[-2.4(0.3) + 1.6(0.18) + 3.6] = 0.3168
\end{aligned}
$$

Generating the remaining entries in a similar manner produces

$$\begin{aligned}
k_{3,x} &= (0.1)g(0.05, 0.267, 0.1584) = 0.54072 \\
k_{3,y} &= (0.1)f(0.05, 0.267, 0.1584) = 0.321264 \\
k_{4,x} &= (0.1)g(0.1, 0.54072, 0.321264) = 0.4800912 \\
k_{4,y} &= (0.1)f(0.1, 0.54072, 0.321264) = 0.28162944
\end{aligned}$$

As a consequence

$$
\begin{aligned}
x_{i+1} &= x_i + \frac{1}{6}\left(k_{1,x} + 2k_{2,x} + 2k_{3,x} + k_{4,x}\right) \\
i_1(0) = x_1 &= x_0 + \frac{1}{6}\left(k_{1,x} + 2k_{2,x} + 2k_{3,x} + k_{4,x}\right) \\
&= 0 + \frac{1}{6}[0.6 + 2(0.534) + 2(0.54072) + 0.4800912] \\
&= 0.5382552 \\
y_{i+1} &= y_i + \frac{1}{6}\left(k_{1,y} + 2k_{2,y} + 2k_{3,y} + k_{4,y}\right) \\
i_2(0) = y_1 &= y_0 + \frac{1}{6}\left(k_{1,y} + 2k_{2,y} + 2k_{3,y} + k_{4,y}\right) \\
&= 0.3196263
\end{aligned}
$$

Exercise

Consider the second order differential equation

$$y'' + y' - 6y = 0$$

with initial conditions

$$y(0) = 1 \text{ and } y'(0) = 0.$$

Then by Runge-Kutta method with step size of $h = 0.25$, find $y(0.75)$.

Predictor-Corrector Methods for System of Differential Equations

Predictor and corrector equations

Adams-bashforth and Adams-moulton methods can be used as a pair to contruct a predictor-corrector method for a system invloving two differential equations as follows:

$$
\begin{aligned}
x_{i+1}^{(p)} &= x_i + \frac{h}{24}(55f(t_i, x_i, y_i) - 59f(t_{i-1}, x_{i-1}, y_{i-1}) \\
&\quad +37f(t_{i-2}, x_{i-2}, y_{i-2}) - 9f(t_{i-3}, x_{i-3}, y_{i-3})), \\
y_{i+1}^{(p)} &= y_i + \frac{h}{24}(55g(t_i, x_i, y_i) - 59g(t_{i-1}, x_{i-1}, y_{i-1}) \\
&\quad +37g(t_{i-2}, x_{i-2}, y_{i-2}) - 9g(t_{i-3}, x_{i-3}, y_{i-3})), \\
x_{i+1}^{(c)} &= x_i + \frac{h}{24}(9f(t_{i+1}, x_{i+1}^{(p)}, y_{i+1}^{(p)}) + 19f(t_i, x_i, y_i) \\
&\quad -5f(t_{i-1}, x_{i-1}, y_{i-1}) + f(t_{i-2}, x_{i-2}, y_{i-2})), \\
y_{i+1}^{(c)} &= y_i + \frac{h}{24}(9g(t_{i+1}, x_{i+1}^{(p)}, y_{i+1}^{(p)}) + 19g(t_i, x_i, y_i) \\
&\quad -5g(t_{i-1}, x_{i-1,i-1}) + g(t_{i-2}, x_{i-2}, y_{i-2})).
\end{aligned}
$$

Example

Use the Adams-Bashforth four-step and the Adams-Moulton methods with step size $h = 0.2$ to find estimates for the solution of

$$\begin{aligned} x'(t) &= 2y(t)\sin t, \qquad x(0) = 0 \\ y'(t) &= x(t) - \sin^2 t - \sin t, \qquad y(0) = 1 \end{aligned}$$

at $t = 0.8$.

Example
Solution

The Runge-Kutta fourth order method can be used to approximate required initial conditions.

The first row of the table came from the initial conditions, and the remaining rows come from the application of the Runge-Kutta fourth-order method. The last two columns use the functions $f(t, x, y) = 2y \sin t$ and $g(t, x, y) = x - \sin^2 t - \sin t$.

i	t_i	x_i	y_i	f_i	g_i
0	0.0	0.00000	1.00000	0.00000	0.00000
1	0.2	0.039483	0.98007	0.38942	-0.19866
2	0.4	0.15167	0.92107	0.71736	-0.38940
3	0.6	0.31885	0.82535	0.93205	-0.56461

Using the Adams-Bashforth four-step method, we get

$$
\begin{aligned}
x_4^{(p)} &= x_3 + \frac{h}{24}(55f_3 - 59f_2 + 37f_1 - 9f_0) \\
&= 0.31885 + \frac{0.2}{24}(55(0.93205) - 59(0.71736) \\
&\quad +37(0.38942) - 9(0)) \\
&= 0.51341 \\
y_4^{(p)} &= y_3 + \frac{h}{24}(55g_3 - 59g_2 + 37g_1 - 9g_0) \\
&= 0.82535 + \frac{0.2}{24}(55(-0.56461) - 59(-0.38940) \\
&\quad +37(-0.19866) - 9(0)) \\
&= 0.69677.
\end{aligned}
$$

Before correcting, we need values of the functions f and g at $t = 0.8$. We'll use our predicted values for this:

$$\begin{aligned} f_4 &= f(t_4, x_4^{(p)}, y_4^{(p)}) = f(0.8, 0.51341, 0.69677) = 0.99966 \\ g_4 &= g(t_4, x_4^{(p)}, y_4^{(p)}) = g(0.8, 0.51341, 0.69677) = -0.71854. \end{aligned}$$

Example

Solution ⇒ Cont...

Correcting with the Adams-Moulton method we get

$$
\begin{aligned}
x_4^{(c)} &= x_3 + \frac{h}{24}(9f_4 + 19f_3 - 5f_2 + f_1) \\
&= 0.31885 + \frac{0.2}{24}(9(0.99966) + 19(0.93205) \\
&\quad -5(0.71736) + 0.38942) \\
&= 0.51476 \\
y_4^{(c)} &= y_3 + \frac{h}{24}(9g_4 + 19g_3 - 5g_2 + g_1) \\
&= 0.82535 + \frac{0.2}{24}(9(-0.71854) + 19(-0.56461) \\
&\quad -5(-0.38940) - 0.19886) \\
&= 0.69663
\end{aligned}
$$

Thus, $x(0.8) \approx 0.51476$ and $y(0.8) \approx 0.69663$.

Analytical Solutions for Higher Order ODE

The Existence and Uniqueness Theorem

Consider the n^{th} order differential equation

$$y^{(n)}(t) = f(t, y(t), y'(t), \cdots, y^{(n-1)}(t)). \qquad (1)$$

Suppose t_0 is a given initial point $t = t_0$, and suppose $a_0, a_1, \cdots, a_{n-1}$ are given constants.

Then there is exactly one solution to the differential equation (1) which satisfies the initial conditions.

$$y(t_0) = a_0, \quad y'(t_0) = a_1, \quad y''(t_0) = a_2, \quad \cdots, \quad y^{(n-1)}(t_0) = a_{n-1}. \quad (2)$$

The general solution

If we solve the differential equation (1), then we will end up with a formula for the solution

$$y = y(t, c_1, c_2, c_3, \cdots, c_n),$$

which contains a number of constants.

Sometimes the way we get the solution leaves the possibility that there might be even more solutions than the ones we found.

If we can show that there actually aren't any other solutions, then our solution is the **general solution**.

The superposition principle

Consider a linear homogeneous equation

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + p_2(t)y^{(n-2)}(t) + \cdots + p_n(t)y(t) = 0. \quad (3)$$

The superposition principle states that

- if $y_1(t)$ and $y_2(t)$ are solutions of (3), then so is $y_1 + y_2$.
- if $y_1(t)$ is a solution to (3), and if c is any constant, then $cy_1(t)$ is also a solution of (3).

The superposition principle implies that if you have n solutions

$$y_1(t), \quad y_2(t), \quad \cdots, \quad y_n(t)$$

then any linear combination

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

is also a solution, as long as the $c_1, \cdots c_n$ are constants.

The general solution and the Wronskian

If $y_1, \cdots, y_n$ are solutions to the homogeneous equation (3), then

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

is the general solution of that equation if and only if

$$W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0.$$

The determinant $W(t_0)$ is called **the Wronskian** of the solutions $y_1, y_2, \cdots, y_n$.

Abel's formula

The Norwegian mathematician Nils Henrik Abel discovered a nice formula which relates the Wronskian $W(t)$ for different values of t.

Abel's formula says

$$W(t_1) = W(t_0)e^{-\int_{t_0}^{t_1} p_1(t)\mathrm{d}t}$$

and he found this by first showing that the Wronskian satisfies a first order differential equation

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = -p_1(t)W(t),$$

known as Abel's differential equation.

Example 1

Suppose you are given

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = \sinh x, \quad y_4(x) = \cosh x,$$

as solutions of the fourth order differential equation $y^{(4)} - y = 0$.

(a) Use the superposition principle to write a solution for the above differential equation.

(b) Could this be the general solution?

(a) The superposition principle tells us that

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sinh x + c_4 \cosh x,$$

is a solution for any choice of the constants $c_1, \cdots, c_4$.

(b) To answer this question we compute the Wronskian

$$
\begin{aligned}
W(x) &= \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ y_1'(x) & y_2'(x) & y_3'(x) & y_4'(x) \\ y_1''(x) & y_2''(x) & y_3''(x) & y_4''(x) \\ y_1'''(x) & y_2'''(x) & y_3'''(x) & y_4'''(x) \end{vmatrix} \\
&= \begin{vmatrix} e^x & e^{-x} & \sinh x & \cosh x \\ e^x & -e^{-x} & \cosh x & \sinh x \\ e^x & e^{-x} & \sinh x & \cosh x \\ e^x & -e^{-x} & \cosh x & \sinh x \end{vmatrix}
\end{aligned}
$$

The first and third rows in this determinant are equal, so the conclusion is $W(x) = 0$.

So the solution is not the general solution.

Example 2

Suppose you are given

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = \sin x, \quad y_4(x) = \cos x$$

as solutions of the fourth order differential equation $y^{(4)} - y = 0$.

(a) Use the superposition principle to write a solution for the above differential equation.

(b) Could this be the general solution?

Example 2
Solution

(a) The superposition principle tells us that

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x,$$

is a solution for any choice of the constants $c_1, \cdots, c_4$.

(b) To answer this question we compute the Wronskian

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) & y_4(x) \\ y_1'(x) & y_2'(x) & y_3'(x) & y_4'(x) \\ y_1(x)'' & y_2''(x) & y_3''(x) & y_4''(x) \\ y_1(x)''' & y_2'''(x) & y_3'''(x) & y_4'''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^x & e^{-x} & \sin x & \cos x \\ e^x & -e^{-x} & \cos x & -\sin x \\ e^x & e^{-x} & -\sin x & -\cos x \\ e^x & -e^{-x} & -\cos x & \sin x \end{vmatrix} \\ &= 8 \end{aligned}$$

This time the Wronskian is not zero, so

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x,$$

is the general solution.

Second Order Linear Differential Equations

Different forms of second order ODEs

1. $y'' + p(t)y' + q(t)y = g(t) \Leftarrow$ second order linear equations
2. $y'' + p(t)y' + q(t)y = 0 \Leftarrow$ second order linear homogeneous equations
3. $ay'' + by' + cy = 0,\ a \neq 0 \Leftarrow$ second order linear homogeneous equations that contain constant coefficients

The Wronskian of two functions y_1 and y_2 is given by the determinant:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

It can be shown that if the Wronskian of two functions is nonzero on an interval then the two functions are **linearly independent** on the interval.

If y_1 and y_2 are two solutions of the equation, $y'' + p(t)y' + q(t)y = 0$, then

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)\ e^{-\int_{t_0}^{t} p(x)\mathrm{d}x}.$$

If $W(y_1, y_2)(t) \neq 0$ for every t then y_1 and y_2 are linear independent.

The set $\{y_1, y_2\}$ is called Fundamental Solution Set.

Then, the general solution y is given by $y = c_1 y_1 + c_2 y_2$.

Linear independence
Cont...

$$W(y_1, y_2)(t) \text{ is nonzero}$$
$$\Updownarrow$$
$$y_1,\ y_2 \text{ are linearly independent}$$
$$\Updownarrow$$
$$y_1,\ y_2 \text{ are Fundamental Solutions Set}$$
$$\Updownarrow$$
$$y = c_1 y_1 + c_2 y_2 \text{ is a general solution of the equation}$$

Example

Let y_1 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 1 \quad y'(0) = -1$$

and y_2 be the solution of the IVP

$$y'' + (2x-1)y' + \sin(e^x)y = 0, \quad y(0) = 2 \quad y'(0) = 1.$$

(a) Find the Wronskian of y_1 and y_2.

(b) Deduce general solution to the above IVP.

Example
Solution

$$
\begin{aligned}
W(y_1, y_2)(x) &= W(y_1, y_2)(x_0)\ e^{-\int_{x_0}^{x} p(t)\mathrm{d}t} \\
W(y_1, y_2)(x_0) &= \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \\
W(y_1, y_2)(x) &= 3\ e^{-\int_{x_0=0}^{x}(2t-1)\mathrm{d}t} = 3e^{-x^2+x} \neq 0
\end{aligned}
$$

$W(y_1, y_2)(x) \neq 0 \Rightarrow y_1$ and y_2 are linearly independent.

$\{y_1, y_2\}$ is the Fundamental Solution Set.

Therefore, general solution is $y = c_1 y_1 + c_2 y_2$.

The characteristic polynomial

Consider the second order linear homogeneous equation:

$$ay'' + by' + cy = 0. \tag{4}$$

Let $y = e^{rt}$ be a solution of (4), for some unknown constant r.

Substitute y, $y' = re^{rt}$, and $y'' = r^2e^{rt}$ into (4), we get

$$\begin{aligned} ay'' + by' + cy &= 0 \\ ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ e^{rt}(ar^2 + br + c) &= 0 \end{aligned}$$

Since e^{rt} is never zero, the above equation is satisfied.

This polynomial, $ar^2 + br + c = 0$, is called the **characteristic polynomial** of the differential equation (4).

Example 1

Find the unique particular solution for the second order differential equation:

$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = -7.$$

Example 1
Solution

The characteristic equation is $r^2 + 5r + 4 = (r + 1)(r + 4) = 0$, the roots of the polynomial are $r = -1$ and -4.

The general solution is then $y = c_1 e^{-t} + c_2 e^{-4t}$.

The values of c_1 and c_2 can be found by solving for c_1 and c_2 using the initial conditions.

The solution is $c_1 = -1$, and $c_2 = 2$.

Therefore, the particular solution is $y = -e^{-t} + 2e^{-4t}$.

Example 2

Find the general solution of

$$y'' + 4y' + 4y = 0.$$

Example 2

Solution

In this case our characteristic equation is $r^2 + 4r + 4 = 0$ which has only one root $r_1 = -2$. And so according to theory in Chapter 2, a fundamental set of solutions are given by $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$, and so the general solution is given by:

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

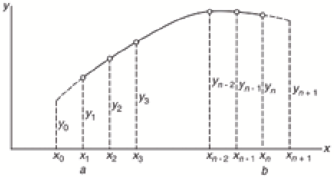
Finite Difference Method for Linear ODE

Initial value problem vs. boundary value problem

- All the initial conditions in an initial value problem must be taken at the same point t_0.
- For example $y(t_0) = y_0$ and $y'(t_0) = y'_0$ at the same point t_0.
- The sets of conditions above where the values are taken at different points are known as **boundary conditions**.
- A boundary value problem where a differential equation is bundled with (two or more) boundary conditions does not have the existence and uniqueness guarantee.

Finite difference method

- In the finite difference method we divide the range of integration (a, b) into $n-1$ equal subintervals of length h each, as shown in Figure.
- The values of the numerical solution at the mesh points are denoted by $y_i, i = 1, 2..., n$; the two points outside (a, b) will be explained shortly.



Finite difference method
Cont...

We then make two approximations:

1. The derivatives of y in the differential equationare replaced by the finite difference expressions. It is common practice to use the first central difference approximations.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \approx \frac{y(t+h)-y(t-h)}{2h} \Leftarrow \text{Central difference}$$

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} \approx \frac{y(t+h)-2y(t)+y(t-h)}{(h)^2} \Leftarrow \text{Central difference}$$

2. The differential equation is enforced only at the mesh points.

Finite difference method
Cont...

After following above two steps, the differential equations will be replaced by n simultaneous algebraic equations, the unknowns being $y_i, i = 1, 2, ...n$.

Consider the second-order differential equation

$$y'' = f(x, y, y')$$

with the boundary conditions

$$\begin{aligned} y(a) &= \alpha \text{ or } y'(a) = \alpha \\ y(b) &= \beta \text{ or } y'(b) = \beta \end{aligned}$$

Approximating the derivatives at the mesh points by finite differences, the problem becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) \quad i = 1, \cdots, n. \qquad (5)$$

$$y_1 = \alpha \text{ or } \frac{y_2 - y_0}{2h} = \alpha \qquad (6)$$

$$y_n = \beta \text{ or } \frac{y_{n+1} - y_{n-1}}{2h} = \beta \qquad (7)$$

Finite difference method
Cont...

Note the presence of y_0 and y_{n+1}, which are associated with points outside the solution domain (a, b). This "spillover" can be eliminated by using the boundary conditions. But before we do that, let us rewrite Eqs. (5) as

$$y_0 - 2y_1 + y_2 - h^2 f\left(x_1, y_1, \frac{y_2 - y_0}{2h}\right) = 0 \quad (8)$$

$$y_{i-1} - 2y_i + y_{i+1} - h^2 f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0 \quad (9)$$

$$\text{for } i = 2, 3, \cdots, n-1$$

$$y_{n-1} - 2y_n + y_{n+1} - h^2 f\left(x_n, y_n, \frac{y_{n+1} - y_{n-1}}{2h}\right) = 0 \quad (10)$$

The boundary conditions on y are easily dealt with: Eq. (8) is simply replaced by $y_1 - \alpha = 0$ and and Eq. (10) is replaced by $y_n - \beta = 0$. If y' are prescribed, we obtain from Eqs. (6,7) $y_0 = y_2 - 2h\alpha$ and $y_{n+1} = y_{n-1} + 2h\beta$, which are then substituted into Eqs. (8) and (10), respectively. Hence we finish up with n equations in the unknowns $y_i, i = 1, 2, \cdots, n$.

Finite difference method
Cont...

$$\begin{aligned} y_1 - \alpha &= 0 \text{ if } y(a) = \alpha \\ -2y_1 + 2y_2 - h^2 f(x_1, y_1, \alpha) - 2h\alpha &= 0 \text{ if } y'(a) = \alpha \end{aligned} \quad (11)$$

$$y_{i-1} - 2y_i + y_{i+1} - h^2 f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0 \quad (12)$$
$$\text{for } i = 2, 3, \cdots, n-1$$

$$\begin{aligned} y_n - \beta &= 0 \text{ if } y(b) = \beta \\ 2y_{n-1} - 2y_n - h^2 f(x_n, y_n, \beta) + 2h\beta &= 0 \text{ if } y'(b) = \beta \end{aligned} \quad (13)$$

Example

Write down equations for the following linear boundary value problem using $n = 11$:

$$y'' = -4y + 4x \qquad y(0) = 0 \qquad y'(\pi/2) = 0.$$

In this case $\alpha = 0$ (applicable to y), $\beta = 0$ (applicable to y') and $f(x, y, y') = -4y + 4x$. Hence equations are

$$\begin{aligned} y_1 &= 0 \\ y_{i-1} - 2y_i + y_{i+1} - h^2(-4y_i + 4x_i) &= 0, \\ \text{for } i = 2, 3, ..., 10 & \\ 2y_{10} - 2y_{11} - h^2(-4y_{11} + 4x_{11}) &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & & & \\ 1 & -2+4h^2 & 1 & & \\ & & & & \\ & & & & \\ & & 1 & -2+4h^2 & 1 \\ & & & 2 & -2+4h^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{10} \\ y_{11} \end{pmatrix} = \begin{pmatrix} 0 \\ 4h^2 x_2 \\ . \\ . \\ . \\ 4h^2 x_{10} \\ 4h^2 x_{11} \end{pmatrix}$$

By solving the system, we can find $y_1, y_2, \cdots, y_{11}$.

Thank you !