Mathematical Modelling-II

(AMT221β/IMT221β)

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Numerical Solutions for Ordinary Differential Equations

Introduction

- Ordinary differential equations occur in many scientific disciplines, for instance in physics, chemistry, biology, and economics.
- But most of those differential equations cannot be solved analytically.
- However, a numeric approximation to the solution is often good enough to solve those differential equations.
- The algorithms we discuss here can be used to compute such numerical approximations.

Analytical vs numerical solutions

- An **analytical solution** of an ODE is a formula $y(t)$, that we can evaluate, differentiate, or analyze in any way we want.

- A **numerical solution** of an ODE is simply a table of absciss and approximate values (t_k, y_k) that approximate the value of an analytical solution.

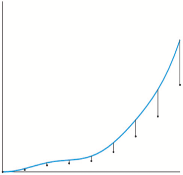

Fig: Analytical vs numerical solutions

Explicit and implicit methods

- Explicit and implicit methods are approaches used in obtaining numerical solutions of time-dependent ordinary and partial differential equations.

- Explicit methods calculate the state of a system at a later time from the state of the system at the current time.

- Mathematically, if $y(t)$ is the current system state and $y(t + \Delta t)$ is the state at the later time (Δt is a small time step) then, for an explicit method

$$y(t + \Delta t) = f(y(t))$$

to find $y(t + \Delta t)$.

- The implicit methods find a solution by solving an equation involving both the current state of the system and the later one.
- For an implicit method one solves an equation

$$g(y(t), y(t+\Delta t)) = 0$$

to find $y(t+\Delta t)$.

Initial value problems

- An initial value problem (IVP) is a differential equation which describes something that changes by specifying an initial state, and giving a rule for how it changes over time.
- **Eg:**

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 5 - t, \quad y(0) = -2.$$

- Thus, a simple IVP would state that at time $t = 0$, the value of y is -2, and that thereafter, y changes according to the rule $\frac{\mathrm{d}y}{\mathrm{d}t} = 5 - t$.

The errors in numerical approximations

- Any approximation of a function necessarily allows a possibility of deviation from the exact value of the function.
- Error is the term used to denote difference between exact solution and numerical approximation.
- Error occurs in an approximation for several reasons.

- In numerical analysis, truncation error is the error made by truncating an infinite sum and approximating it by a finite sum.

- For instance,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots$$

- If we approximate the sine function by the first two non-zero term of its Taylor series, as in $\sin(x) \approx x - \frac{1}{6}x^3$ for small x, the resulting error is a truncation error.

- The only way to completely avoid truncation error is to use exact calculations.

- The local truncation error of a numerical method is error made in a single step.
- It is the difference between the numerical solution after one step, y_1, and the exact solution at time $t_1 = t_0 + h$.

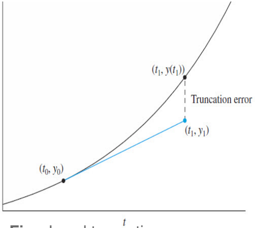

Fig: Local truncation error

- Global truncation error is the amount of truncation error that occurs in the use of a numerical approximation to solve a problem.
- The global truncation error is the cumulative effect of the local truncation errors committed in each step.

Euler's Method

About Euler's method

- The Euler's method is the most basic explicit method.
- It was named after Leonhard Euler.
- It is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value.

- The Euler's method is used to obtain an approximation to the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

- The approximations to $y(t)$ will be generated at various values, called **mesh points**, in the interval $[a, b]$.

Set up an equally-distributed mesh

- We first make the stipulation that the mesh points are equally distributed throughout the interval $[a, b]$.
- This condition is ensured by choosing a positive integer N and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, ..., N.$$

- The common distance between the points

$$h = (b - a)/N = t_{i+1} - t_i$$

is called the **step size.**

Use Taylor's theorem to derive Euler's method

Suppose that $y(t)$, the unique solution to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \;\; a \le t \le b, \;\; y(a) = \alpha$$

has two continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, ..., N - 1$,

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2} y''(\xi_i)$$

for some number ξ_i in (t_i, t_{i+1}).

Use Taylor's theorem to derive Euler's method
Cont...

Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)$$

and, because $y(t)$ satisfies the differential equation $y' = f(t, y)$, we write

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i).$$

Euler's method

Euler's method constructs $y_i \approx y(t_i)$, for each $i = 1, 2, ..., N$, by deleting the remainder term. Thus Euler's method is

$$\begin{aligned} y_0 &= \alpha \\ y_{i+1} &= y_i + hf(t_i, y_i), \quad \text{for each } i = 0, 1, ..., N-1. \end{aligned}$$

Example 1

Using Euler's method, obtain the solution of

$$y' = x - y \text{ with } y(0) = 1,$$

at $x = 0.6$. Use a step size of $h = 0.2$.

Here $f(x, y) = x - y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$. We have to find out the values of y at $x_1 = 0.2$, $x_2 = 0.4$ and $x_3 = 0.6$.

Now, $f(x_0, y_0) = f(0, 1) = 0 - 1 = -1$.

By Euler's method, we have

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (0.2)(-1) \\ &= 0.8 \end{aligned}$$

For the next step,

$$\begin{aligned} f(x_1, y_1) &= f(0.2, 0.8) \\ &= 0.2 - 0.8 \\ &= -0.6 \end{aligned}$$

Therefore

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.8 + (0.2)(-0.6) \\ &= 0.68 \end{aligned}$$

We get, for $x_2 = 0.4$,

$$\begin{aligned} f(x_2, y_2) &= f(0.4, 0.68) \\ &= -0.28 \end{aligned}$$

Therefore

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.68 + (0.2)(-0.28) \\ &= 0.624 \end{aligned}$$

Now, tabulating the solution, we have

x	0	0.2	0.4	0.6
y	1	0.8	0.68	0.624

Example 2

Compute the first four steps in the Eulers method approximation to the solution of

$$y' = y^{1/2}t^2,$$

with $y(0) = 1$, using the step size $h = 0.5$. Compare the results with the actual solution to the initial value problem.

Here $f(t,y) = y^{1/2}t^2$, $t_0 = 0$, $y_0 = 1$ and $h = 0.5$. We have to find out the values of y at $t_1 = 0.5$, $t_2 = 1.5$ and $t_3 = 2.0$.

By Euler's method, we have

$$
\begin{aligned}
y_1 &= y_0 + hf(t_0, y_0) \\
y_1 &= y_0 + hf(0, 1) \\
&= 1 + (0.5)1^{1/2}0^2 \\
&= 1
\end{aligned}
$$

Example 2

Solution⇒Cont...

For the next step,

$$
\begin{aligned}
y_2 &= y_1 + hf(t_1, y_1)\\
y_2 &= 1 + (0.5)f(0.5, 1)\\
&= 1.1250
\end{aligned}
$$

For the next step,

$$
\begin{aligned}
y_3 &= y_2 + hf(t_2, y_2)\\
y_3 &= 1.1250 + (0.5)f(1, 1.1250)\\
&= 1.6553
\end{aligned}
$$

For the next step,

$$\begin{aligned} y_4 &= y_3 + hf(t_3, y_3) \\ y_4 &= 1.6553 + (0.5)f(1.5, 1.6553) \\ &= 3.1027 \end{aligned}$$

Example 2

Solution⇒Cont...

The DE has exact solution $y(t) = \dfrac{(t^3/3 + 2)^2}{4}$. Computing the solution at the discrete t-values 0, 0.5, 1, 1.5, and 2 we have the following table of comparisons between the approximate and exact values as well as error values.

t_k	Approx y_k	Exact y_k	Error
0	1	1	0
0.5	1	1.1289	0.1289
1.0	1.125	2.2500	1.1250
1.5	1.6553	7.2227	5.5674
2.0	3.1027	25.0000	21.8973

Local truncation error for Euler's method

The local truncation error of the Euler's method is the difference between the numerical solution after one step, y_1, and the exact solution at time $t_1 = t_0 + h$. The numerical solution is given by

$$y_1 = y_0 + hf(t_0, y_0).$$

For the exact solution, we use the Taylor expansion

$$y(t_0 + h) = y(t_0) + hy'(t_0) + \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

Local truncation error for Euler's method
Cont...

The local truncation error (LTE) introduced by the Euler's method is given by the difference between these equations:

$$\mathrm{LTE} = y(t_0 + h) - y_1 = \frac{1}{2}h^2 y''(t_0) + O(h^3).$$

This result is valid if y has a bounded third derivative.

This shows that for small h, the local truncation error is approximately proportional to h^2.

This makes the Euler's method less accurate (for small h) than other higher-order techniques which we shall discuss later in this Chapter.

Global truncation error for Euler's method

- The global truncation error is the error at a fixed time t, after however many steps the methods needs to take to reach that time from the initial time.
- The number of steps is easily determined to be $(t - t_0)/h$, which is proportional to $1/h$, and the error committed in each step is proportional to h^2.
- Thus, it is to be expected that the global truncation error will be proportional to h.

Remark 1

- The Euler's method is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size.

Remark 2

- For any method, the Global Truncation Error (GTE) is one power lower in h than the Local Truncation Error (LTE).
- We have shown that the LTE of Euler's method is $O(h^2)$.
- So the GTE for Euler's method should be $O(h)$.
- By the **order of a method**, we mean the power of h in the GTE.
- Thus the Euler's method is a first order method.

Modified Euler Method

About Modified Euler method

- The Euler's method can easily be implemented.
- But its accuracy is very low.
- So an improvement over this is to take the arithmetic average of the slopes at t_i and t_{i+1} (that is, at the end points of each sub-interval).
- The scheme so obtained is called **Modified Euler method** and its order is two ($O(h^2)$).
- It works first by approximating a value to y_{i+1} and then improving it by making use of average slope.

About Modified Euler method
Cont...

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{2}(y_i' + y_{i+1}') \\
&= y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, y_{i+1}))
\end{aligned}
$$

If Euler's method is used to find the first approximation of y_{i+1} then

$$y_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, y_i + hf(t_i, y_i))).$$

Example

(a) Use the Modified Euler method to obtain approximations to the solution of the intial-value problem

$$y' = y - t^2 + 1, \;\; 0 \leq t \leq 2, \;\; y(0) = 0.5$$

with $h = 0.2$.

(b) If the exact solution is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, then calculate error in each step.

Example
Solution

$$
\begin{aligned}
y_1 &= y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_1)) \\
y_1 &= y_0 + \frac{h}{2}(f(t_0, y_0) + f(t_1, y_0 + hf(t_0, y_0))) \\
&= 0.5 + 0.1(f(0, 0.5) + f(0.2, 0.5 + 0.2(0.5 - 0 + 1)) \\
&= 0.5 + 0.1(f(0, 0.5) + f(0.2, 0, 8)) \\
&= 0.5 + 0.1((0.5 - 0 + 1) + (0.8 - 0.2^2 + 1)) \\
&= 0.826 \\
y_2 &= y_1 + \frac{h}{2}(f(t_1, y_1) + f(t_2, y_2)) \\
y_2 &= y_1 + \frac{h}{2}(f(t_1, y_1) + f(t_2, y_1 + hf(t_1, y_1))) \\
&= 1.2069
\end{aligned}
$$

Example

Solution⇒Cont...

Tabulating the solution, we have

t_i	Exact $y(t_i)$	Modified Euler Method (y_i)	Error $\lvert y(t_i) - y_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8292986	0.8260000	0.0032986
0.4	1.2140877	1.2069200	0.0071677
0.6	1.6489406	1.6372424	0.0116982
0.8	2.1272295	2.1102357	0.0169938
1.0	2.6408591	2.6176876	0.0231715
.	.	.	.
.	.	.	.
.	.	.	.
2.0	5.3054720	5.2330546	0.0724173

Heun's Method

About Heun's method

- Heun's method is a second order ($O(h^2)$) numerical procedure for solving ordinary differential equations (ODEs) with a given initial value.
- It is named after Karl Heun.

About Heun's method
Cont...

The difference equation for the method is:

$$
\begin{aligned}
y_0 &= \alpha \\
y_{i+1} &= y_i + \frac{h}{4}\left(f(t_i, y_i) + 3f\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}hf(t_i, y_i)\right)\right) \\
&\qquad\qquad \text{for each } i = 0, 1, 2, ..., N-1.
\end{aligned}
$$

Example

(a) Use the Heun's method to obtain approximations to the solution of the intial-value problem

$$y' = y - t^2 + 1, \;\; 0 \leq t \leq 2, \;\; y(0) = 0.5$$

with $h = 0.2$.

(b) If the exact solution is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, then calculate error in each step.

Example
Solution

$$
\begin{aligned}
h &= 0.2 \quad t_0 = 0 \quad y_0 = 0.5 \\
y_1 &= y_0 + \frac{h}{4}(f(t_0, y_0) + 3f(t_0 + \frac{2}{3}h, y_0 + \frac{2}{3}hf(t_0, y_0))) \\
y_1 &= 0.8273 \\
y_2 &= y_1 + \frac{h}{4}(f(t_1, y_1) + 3f(t_1 + \frac{2}{3}h, y_1 + \frac{2}{3}hf(t_1, y_1))) \\
y_2 &= 1.2098
\end{aligned}
$$

Tabulating the solution, we have

t_i	Exact $y(t_i)$	Heun's Method (y_i)	Error $\lvert y(t_i) - y_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8292986	0.8267333	0.0019653
0.4	1.2140877	1.2098800	0.0042077
0.6	1.6489406	1.6421869	0.0067537
0.8	2.1272295	2.1176014	0.0096281
1.0	2.6408591	2.6280070	0.0128521
.	.	.	.
.	.	.	.
.	.	.	.
2.0	5.3054720	5.2712645	0.0342074

Runge-Kutta Methods

About Runge-Kutta methods

- The Runge-Kutta methods are an important family of implicit and explicit iterative methods for the approximation of solutions of ordinary differential equations.
- These techniques were developed by the German mathematicians C. Runge and M. W. Kutta.

About Runge-Kutta methods
Cont...

- Both Modified Euler method and Heun's method can be seen as extensions of the Euler method into two-stage second-order Runge-Kutta methods.
- The Runge-Kutta method of order three is not generally used.
- The most common Runge-Kutta method in use is of order four.

Fourth order Runge-Kutta method

The difference equation form is:

$$
\begin{aligned}
y_0 &= \alpha \\
k_1 &= hf(t_i, y_i) \\
k_2 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_1\right) \\
k_3 &= hf\left(t_i + \frac{h}{2}, y_i + \frac{1}{2}k_2\right) \\
k_4 &= hf(t_{i+1}, y_i + k_3) \\
y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&\qquad\qquad \text{for each } i = 0, 1, 2, ..., N-1.
\end{aligned}
$$

Remark

- The method has local truncation error ($O(h^5)$).
- The reason for introducing the notaton k_1, k_2, k_3, k_4 into the method is to eliminate the need for successive nesting in the second variable of $f(t, y)$.

Example

(a) Use the Runge-Kutta method of order four to obtain approximations to the solution of the intial-value problem

$$y' = y - t^2 + 1, \;\; 0 \leq t \leq 2, \;\; y(0) = 0.5$$

with $h = 0.2$.

(b) If the exact solution is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, then calculate error in each step.

Example
Solution

$$
\begin{aligned}
t_0 &= 0 \quad y_0 = 0.5 \\
k_1 &= hf(t_0, y_0) \\
&= 0.2f(0, 0.5) = 0.3 \\
k_2 &= hf\left(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) \\
&= 0.2f\left(0 + \frac{0.2}{2}, 0.5 + \frac{1}{2}0.3\right) = 0.328 \\
k_3 &= hf\left(t_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) \\
&= 0.2f\left(0 + \frac{0.2}{2}, 0.5 + \frac{1}{2}0.328\right) = 0.3308 \\
k_4 &= hf(t_1, y_0 + k_3) = 0.2f(0.2, 0.5 + 0.3308) = 0.35816
\end{aligned}
$$

Example

Solution⇒

$$
\begin{aligned}
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&= 0.5 + \frac{1}{6}(0.3 + 2 \times 0.328 + 2 \times 0.3308 + 0.35816) \\
&= 0.8292933
\end{aligned}
$$

Tabulating the solution, we have

t_i	Exact $y(t_i)$	Runge-Kutta method (y_i)	Error $\lvert y(t_i) - y_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
.	.	.	.
.	.	.	.
.	.	.	.
2.0	5.3054720	5.3053630	0.0001089

Remark 1

- The main computational effort in applying the Runge-Kutta methods is the evaluation of f.
- In the second-order methods, the cost is two functional evaluations per step.
- The Runge-Kutta method of order four requires 4 evaluations per step.
- This is why the less order methods with smaller step size are used in preference to the higher order methods using a larger step size.

Remark 2

- The Runge-Kutta methods of order four requires four evaluations per step, so it should give more accurate answer than Euler's method with one-fourth the step size if it is to be superior.

- Similarly, if the Runge-Kutta method of order four is to be superior to the second-order Runge-Kutta methods, it should give more accuracy with step size h than a second-order method with step size $\frac{1}{2}h$, because the fourth-order method requires twice as many evaluations per step.

- An illustration of the superiority of the Runge-Kutta fourth-order method by this measure is shown in the following example.

Example

For the problem

$$y' = y - t^2 + 1. \ \ 0 \leq t \leq 2, \ \ y(0) = 5.$$

Compare Euler's method with $h = 0.025$, the Modified Euler method with $h = 0.05$, and Runge-Kutta method with $h = 0.1$ at the common mesh points of these methods 0.1, 0.2, 0.3, 0.4 and 0.5.

Example
Solution

t_i	Exact $y(t_i)$	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta order four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
.	.	.	.	.
.	.	.	.	.
0.5	1.4256394	1.4147264	1.4250141	1.4256384

Picard's Method

About Picard's method

- There are some differential equation which cannot be solved by one of the standard methods known so far.
- Picard's iteration is a constructive procedure for establishing the existence of a solution to a differential equation $y' = f(t, y)$ that passes through the point (t_0, y_0).
- Picard's method converts the differential equation into an equation involving integrals, which is called an integral equation.

Integral equation in Picard's method

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y), \quad y(t_0) = y_0. \tag{1}$$

Integrating this equation, we get

$$\begin{aligned}
\mathrm{d}y &= f(t,y)\mathrm{d}t \\
\int_{t_0}^{t} \mathrm{d}y &= \int_{t_0}^{t} f(t,y)\mathrm{d}t \\
y(t) - y(t_0) &= \int_{t_0}^{t} f(t,y)\mathrm{d}t \\
y(t) &= y(t_0) + \int_{t_0}^{t} f(t,y)\mathrm{d}t = y_0 + \int_{t_0}^{t} f(t,y)\mathrm{d}t \quad (2)
\end{aligned}$$

This is an integral equation and hence the problem of solving differential equation (1) has been reduced to solve the integral equation (2)

Integral equation in Picard's method

Cont...

- Since the information concerning the expression of y in terms of t is absent, the integral or the R.H.S. of (2) cannot be evaluated.
- Hence the exact value of y cannot be obtained.
- Therefore we determine a sequence of approximations to the solution integral on the right of (2).

Integral equation in Picard's method
Cont...

For the first approximation $y_1(t)$ to the solution, we put $y = y_0$ in $f(t, y)$ and obtain

$$y_1(t) = y_0 + \int_{t_0}^{t} f(t, y_0)\mathrm{d}t.$$

Similarly

$$\begin{aligned}
y_2(t) &= y_0 + \int_{t_0}^{t} f(t, y_1)\mathrm{d}t \\
y_3(t) &= y_0 + \int_{t_0}^{t} f(t, y_2)\mathrm{d}t \\
y_4(t) &= y_0 + \int_{t_0}^{t} f(t, y_3)\mathrm{d}t
\end{aligned}$$

and so on.

Integral equation in Picard's method
Cont...

Proceeding in this way, the $(n+1)^{\mathrm{th}}$ approximation $y_{n+1}(t)$ is given by

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(t, y_n)\mathrm{d}t.$$

Remark

- Picard's method gives us a sequence of approximations $y_1, y_2, y_3, ..., y_n$.
- This sequence $\{y_n\}n = 1, 2, 3, ...$ converges to exact solution provided that the function $f(t, y)$ is bounded in some region in the neighborhood of (t_0, y_0) and satisfies the **Lipschits condition** namely **there exists a constant** K **such that** $|f(t, y) - f(t, \overline{y})| \leq K|y - \overline{y}|$**, for all** t**.**

Example 1

Apply Picard's method to solve the following initial value problem up to third approximation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 2t^2 - 3, \quad y(0) = 2.$$

Example 1
Solution

$$t_0 = 0 \quad y_0 = y(0) = 2 \quad f(t,y) = \frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 2t^2 - 3$$

Therefore by Picard's method we get

$$\begin{aligned}
y_1(t) &= y_0 + \int_{t_0}^{t} f(t, y_0)\mathrm{d}t \\
y_1(t) &= 2 + \int_0^t (2y_0 - 2t^2 - 3)\mathrm{d}t \\
&= 2 + \int_0^t (4 - 2t^2 - 3)\mathrm{d}t \\
&= 2 + \left[t - \frac{2}{3}t^3\right]_0^t = 2 - \frac{2}{3}t^3 + t.
\end{aligned}$$

Thus first approximation is $y_1(t) = y_1 = 2 - \frac{2}{3}t^3 + t$.

The second approximation is

$$
\begin{aligned}
y_2(t) &= y_0 + \int_{t_0}^{t} f(t, y_1)\,\mathrm{d}t \\
y_2(t) &= 2 + \int_{0}^{t} \left[2\left(2 - \frac{2}{3}t^3 + t\right) - 2t^2 - 3\right]\mathrm{d}t \\
&= 2 + t + t^2 - \frac{2}{3}t^3 - \frac{t^4}{4}.
\end{aligned}
$$

Third approximation is

$$
\begin{aligned}
y_3(t) &= y_0 + \int_{t_0}^{t} f(t, y_2)\mathrm{d}t \\
&= 2 + \int_0^t \left[2\left(2 + t + t^2 - \frac{2}{3}t^3 - \frac{t^4}{4}\right) - 2t^2 - 3 \right] \mathrm{d}t \\
&= 2 + t + t^2 - \frac{t^4}{3} - \frac{t^5}{15}.
\end{aligned}
$$

Example 2

Find $y(0.1)$ by Picard's method from the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y-t}{y+t}, \quad y(0) = 1.$$

Example 2
Solution

$$t_0 = 0 \quad y_0 = y(0) = 1 \quad f(t,y) = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y-t}{y+t}$$

Therefore by Picard's method we get

$$\begin{aligned}
y_1(t) &= y_0 + \int_{t_0}^{t} f(t, y_0)\mathrm{d}t \\
y_1(t) &= 1 + \int_0^t \frac{1-t}{1+t}\mathrm{d}t \\
&= 1 + \int_0^t \left(-1 + \frac{2}{1+t}\right)\mathrm{d}t \\
&= 1 - t + 2\log(1+t).
\end{aligned}$$

Second approximation is

$$
\begin{aligned}
y_2(t) &= y_0 + \int_{t_0}^{t} f(t, y_1)\mathrm{d}t \\
&= 1 + \int_0^t \frac{1 - t + 2\log(1+t) - t}{1 - t + 2\log(1+t) + t}\mathrm{d}t \\
&= 1 + \int_0^t \left(1 - \frac{2t}{1 + 2\log(1+t)}\right)\mathrm{d}t
\end{aligned}
$$

which is quite difficult to integrate. Hence using $y_1(t)$ and taking $t = 0.1$ we get

$$
y_1(0.1) = 1 - (0.1) + 2\log(1.1) = 0.9828.
$$

Taylor's Series Method

Motivative example

- How your calculator gives answer for $\sin t$ for any particular value of t that you request?
- It can not remember sin value for every t, because this requires more memory.
- So it uses a polynomial approximation for that.

$$
\begin{aligned}
f'(t_0) &\approx \frac{f(t) - f(t_0)}{(t - t_0)} \\
f(t) &\approx f(t_0) + f'(t_0)(t - t_0)
\end{aligned}
$$

For example $t = 0.2 \Rightarrow$

$$
\begin{aligned}
\sin(0.2) &\approx \sin 0 + \cos 0(0.2 - 0) \\
&\approx 0.2
\end{aligned}
$$

We can obtain a better result using higher order Taylor polynomials.

Approximating a function using Taylor series

Recall that the n^{th} order Taylor expansion of a (smooth) function $f(t)$ about the point $t = t_0$ is the degree n polynomial defined by

$$\begin{aligned} T_n(t) &= \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(t_0)(t - t_0)^i \\ &= f(t_0) + f'(t_0)(t - t_0) + \frac{1}{2} f''(t_0)(t - t_0)^2 \\ &\quad + \frac{1}{6} f'''(t_0)(t - t_0)^3 + \ldots \end{aligned}$$

Such expansions are extremely useful in that they can be used as approximate expressions for the original function f.

Taylor's theorem says

$$f(t) = T_n(t) + O(|t - t_0|^{n+1})$$

and that moreover

$$f(t) = \lim_{n\to\infty} T_n(t).$$

Therefore, one way to get an approximate solution of a differential equation would be to figure out what its Taylor series looks like and next we talk about it.

Approximating solution of a differential equation

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), \; y(t_0) = y_0. \tag{3}$$

Let $y = y(t)$ be a continuously diffrentiable function satisfying the equation (3).

Expanding y in terms of Taylor's series around the point $t = t_0$, we get

$$y = y_0 + (t - t_0)y_0' + \frac{(t - t_0)^2}{2!}y_0'' + \frac{(t - t_0)^3}{3!}y_0''' + \ldots \tag{4}$$

The value of the differential coefficients $y_0', y_0'', y_0''', \ldots$ at $t = t_0$ can be computed from the equation as follows:

$$y' = f(t, y).$$

Differentiating we get

$$y'' = f_t + f_y \frac{\mathrm{d}y}{\mathrm{d}t} = f_t + ff_y.$$

Differentiating again, we get

$$\begin{aligned} y''' &= f_{tt} + ff_{ty} + ff_{ty} + f^2 f_{yy} + f_t f_y + ff_y^2 \\ &= f_{tt} + 2ff_{ty} + f_t f_y + ff_y^2 + f^2 f_{yy} \end{aligned}$$

and so on differentiating successively, we get y^{iv}, y^{v} etc.

From above equations, we can obtain the values of $y_0', y_0'', y_0''', ...$ at (t_0, y_0) and when used in equation (4) gives us the desired solution.

This works well so long as the successive derivaties can be calculated easily.

Example 1

Use Taylor's series method to find $y(0.1)$ and $y(0.2)$ from the equation:

$$y' = y^2 + x, \quad y(0) = 1.$$

Example 1
Solution

The given differential equation is

$$y' = y^2 + x, \quad y(0) = 1.$$

Here $f(x,y) = y^2 + x, \;\; x_0 = 0, \;\; y_0 = 1.$

The Taylor's series solution is

$$\begin{aligned} y &= y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \frac{(x-x_0)^4}{4!}y_0^{iv} \\ &\quad + \frac{(x-x_0)^5}{5!}y_0^{v} + \ldots \\ y &= y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v}\ldots \end{aligned}$$

To get the solution we should know $y_0', y_0'', y_0''', y_0^{iv}$ and y_0^{v}.

Therefore we need to calculate followings:

$$
\begin{aligned}
y' &= y^2 + x \\
y'' &= 2yy' + 1 \\
y''' &= 2(y')^2 + 2yy'' \\
y^{iv} &= 6y'y'' + 2yy''' \\
y^{v} &= 2yy^{(iv)} + 8y'y''' + 6(y'')^2
\end{aligned}
$$

Example 1
Solution⇒Cont...

Therefore we have to consider y', y'', y''', y^{iv} and y^{v} at (x_0, y_0) to get y_0', y_0'', y_0''' and y_0^{iv}.

$$\begin{aligned}
y' &= y^2 + x &\Rightarrow\quad & y_0' = 1^2 + 0 = 1.0\\
y'' &= 2yy' + 1 &\Rightarrow\quad & y_0'' = 2\times 1\times 1 + 1 = 3\\
y''' &= 2(y')^2 + 2yy'' &\Rightarrow\quad & y_0''' = 2(1)^2 + 2\times 1\times 3 = 8\\
y^{iv} &= 6y'y'' + 2yy'' &\Rightarrow\quad & y_0^{iv} = 6\times 1\times 3 + 2\times 1\times 8 = 34\\
y^{v} &= 2yy^{(iv)} + 8y'y''' + 6(y'')^2 &\Rightarrow\quad & y_0^{(v)} = 186
\end{aligned}$$

Thus the Taylor's series solution is

$$\begin{aligned} y &= y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + \frac{x^4}{4!}y_0^{iv} + \frac{x^5}{5!}y_0^{v}... \\ &= 1 + x.1 + \frac{x^2}{2!}.3 + \frac{x^3}{3!}.8 + \frac{x^4}{4!}.34 + \frac{x^5}{5!}.186... \\ &= 1 + x + \frac{3x^2}{2} + \frac{4}{3}x^3 + \frac{17}{12}x^4 + \frac{31}{20}x^5 + ... \end{aligned}$$

Here $y(0.1) = 1.11647$ and $y(0.2) = 1.27296$.

Example 2

Find by Taylor's series method, the value of y at $x = 0.1$ and $x = 0.2$ to five decimal places from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y - 1, y(0) = 1.$$

Here $y_0 = 1$ and $y' = x^2y - 1$.

Differentiating successively and substituting, we get

$$\begin{aligned} y' = x^2y - 1 \quad &\Rightarrow \quad y'_0 = -1 \\ y'' = 2xy + x^2y' \quad &\Rightarrow \quad y''_0 = 0 \\ y''' = 2y + 4xy' + x^2y'' \quad &\Rightarrow \quad y'''_0 = 2 \\ y^{iv} = 6y' + 6xy''' + x^2y''' \quad &\Rightarrow \quad y^{iv}_0 = -6 \end{aligned}$$

and so on.

Thus by Taylor's series method, we have

$$\begin{aligned} y &= 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + ... \\ &= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + ... \end{aligned}$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Exercise

Solve the following differential equation using Taylors series expansion:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = 3u^2 v, \quad v(1) = 1.$$

Answer:

$$v(u) = 1 + (u - 1)(3) + 7.5(u - 1)^2 + 14.5(u - 1)^3.$$

Multistep Methods

One step methods

- Up to now, all methods we studied were **one step** methods.
- One step method uses information from only one of the previous mesh point, t_i for the approximation for the mesh point t_{i+1}.
- Although these methods might use functional evaluation information at points between t_i and t_{i+1}, they do not retain that information for direct use in future approximations.
- All the information used by these methods is obtained within the subinterval over which the solution is being approximated.

Moving to multistep methods

Since the approximate solution is available at each of the mesh points $t_0, t_1, t_2, ..., t_i$ before the approximation at t_{i+1} is obtained, and because the error $|y_j - y(t_j)|$ tends to increase with j, it seems reasonable to develop methods that use these more accurate previous data when approximating the solution at t_{i+1}.

Multistep methods

Methods using the approximation at more than one previous mesh point to determine the approximation at the next point are called **multistep** methods.

An m**-step multistep method** for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has a difference equation for finding the approximation y_{i+1} at mesh point t_{i+1} represented by the following equation, where m is an integer greater than 1:

$$\begin{aligned} y_{i+1} &= a_{m-1}y_i + a_{m-2}y_{i-1} + ... + a_0 y_{i+1-m} \\ &\quad + h[b_m f(t_{i+1}, y_{i+1}) + b_{m-1} f(t_i, y_i) \\ &\quad + ... + b_0 f(t_{i+1-m}, y_{i+1-m})], \end{aligned} \tag{5}$$

for $i = m-1, m, ..., N-1$, where $h = (b-a)/N$, the $a_0, a_1, ..., a_{m-1}$ and $b_0, b_1, ..., b_m$ are constants, and the starting values $y_0 = \alpha, \;\; y_1 = \alpha_1, \;\; y_2 = \alpha_2, ..., y_{m-1} = \alpha_{m-1}$ are specified.

Multistep methods

Remark 1

- The coefficients $a_0, \ldots, a_{m-1}$ and $b_0, \ldots, b_m$ determine the method.
- The designer of the method chooses the coefficients, balancing the need to get a good approximation to the true solution against the desire to get a method that is easy to apply.
- Often, many coefficients are zero to simplify the method.

Multistep methods
Remark 2

- When $b_m = 0$ the method is called **explicit**, since (5) gives y_{i+1} explicitly in terms of previously determined values.
- When $b_m \neq 0$ the method is called **implicit** , since y_{i+1} occurs on both sides of (5) and is specified only implicitly.

Families of multistep methods

Three families of linear multistep methods are commonly used:

- Adams-Bashforth methods,
- Adams-Moulton methods,
- and the backward differentiation formulas (BDFs).

Adams-Bashforth methods

- The Adams-Bashforth methods are explicit methods.
- The coefficients are $a_{m-1} = 1$ and $a_{m-2} = \cdots = a_0 = 0$, while the b_j are chosen such that the methods has order m (this determines the methods uniquely).

Adams-Bashforth methods
Cont...

Members of Adams-Bashforth family can be written down as follows:

$$
\begin{aligned}
y_{i+1} &= y_i + hf(t_i, y_i) && (6)\\
y_{i+1} &= y_i + h\left(\frac{3}{2}f(t_i, y_i) - \frac{1}{2}f(t_{i-1}, y_{i-1})\right) && (7)\\
y_{i+1} &= y_i + h\left(\frac{23}{12}f(t_i, y_i) - \frac{4}{3}f(t_{i-1}, y_{i-1})\right.\\
&\quad \left. + \frac{5}{12}f(t_{i-2}, y_{i-2})\right) && (8)\\
y_{i+1} &= y_i + \frac{h}{24}\left(55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1})\right.\\
&\quad \left. + 37f(t_{i-2}, y_{i-2}) - 9f(t_{i-3}, y_{i-3})\right) && (9)
\end{aligned}
$$

Fourth-order Adams-Bashforth method

The equation (9) with

$$
\begin{aligned}
y_0 &= \alpha, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad y_3 = \alpha_3, \\
y_{i+1} &= y_i + \frac{h}{24}\Big(55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) \\
&\quad -9f(t_{i-3}, y_{i-3})\Big),
\end{aligned}
$$

for each $i = 3, 4, ..., N-1$, define an explicit four-step method known as the **fourth-order Adams-Bashforth** method.

Fourth-order Adams-Bashforth method

Note

- We cannot calculate the first three steps by this method. So we apply another method for first three steps.
- Usually Runge-Kutta or Taylor method is used to generate first three steps values.

Example 1

Use the Runge-Kutta method of order 4 with $h = 0.2$ to approximate the solutions to the initial value problem for $y(0.2)$, $y(0.4)$ and $y(0.6)$.

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

Use above approximations as starting values for the 4th-order Adams-Bashforth method to compute new approximations for $y(0.8)$ and $y(1.0)$, and compare these new approximations to those produced by the Runge-Kutta method of order 4.

The first four approximations were found to be $y(0) = y_0 = 0.5$; $y(0.2) \approx y_1 = 0.8292933$; $y(0.4) \approx y_2 = 1.2140762$; and $y(0.6) \approx y_3 = 1.6489220$.

Now we can use above approximations as starting values for the 4th-order Adams-Bashforth method to compute new approximations for $y(0.8)$ and $y(1.0)$.

For the 4th-order Adams-Bashforth, we have

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{24}(55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) \\
&\quad -9f(t_{i-3}, y_{i-3})), \\
y_4 &= y_3 + \frac{0.2}{24}(55f(t_3, y_3) - 59f(t_2, y_2) + 37f(t_1, y_1) \\
&\quad -9f(t_0, y_0)), \\
y_4 &= 1.6489220 + \frac{0.2}{24}(55f(0.6, 1.6489220) - 59f(0.4, 1.2140762) \\
&\quad +37f(0.2, 0.8292933) - 9f(0, 0.5)), \\
&= 2.1272892
\end{aligned}
$$

For the 4th-order Adams-Bashforth, we have

$$\begin{aligned} y_{i+1} &= y_i + \frac{h}{24}(55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) \\ &\quad -9f(t_{i-3}, y_{i-3})), \\ y_5 &= y_4 + \frac{0.2}{24}(55f(t_4, y_4) - 59f(t_3, y_3) + 37f(t_2, y_2) \\ &\quad -9f(t_1, y_1)), \\ y_5 &= 2.1272892 + \frac{0.2}{24}(55f(0.8, 2.1272892) - 59f(0.6, 1.6489220) \\ &\quad +37f(0.4, 1.2140762) - 9f(0.2, 0.8292933)), \\ &= 2.6410533 \end{aligned}$$

The error for these approximations at $t = 0.8$ and $t = 1.0$ are, respectively:

$$\begin{aligned} |2.1272295 - 2.1272892| &= 5.97 \times 10^{-5} \text{ and} \\ |2.6410533 - 2.6408591| &= 1.94 \times 10^{-4} \end{aligned}$$

The corresponding Runge-Kutta approximations had errors:

$$\begin{aligned} |2.1272027 - 2.1272892| &= 2.69 \times 10^{-5} \text{and} \\ |2.6408227 - 2.6408591| &= 3.64 \times 10^{-5} \end{aligned}$$

Example 2

Use 4th-order Adams-Bashforth method to find $y(0.8)$ and $y(1.0)$ given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2, \quad y(0) = 0.$$

We take $h = 0.2$ with $x_0 = 0$, $y_0 = 0$ and use Runge-Kutta fourth order method to obtain

$$
\begin{aligned}
k_1 &= hf(x_0, y_0) = 0.2 \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = 0.202 \\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = 0.20204 \\
k_4 &= hf(x_1, y_0 + k_3) = 0.20816 \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027
\end{aligned}
$$

By following the same procedure, we can obtain y_2, and y_3.

Thus we have

x_i	y_i
$x_0 = 0.0$	$y_0 = 0.0000$
$x_1 = 0.2$	$y_1 = 0.2027$
$x_2 = 0.4$	$y_2 = 0.4228$
$x_3 = 0.6$	$y_3 = 0.6841$

Example 2

Solution $\Rightarrow$ Cont...

x_i	y_i	$f(x_i, y_i) = 1 + y_i^2$
$x_0 = 0.0$	$y_0 = 0.0000$	$f(x_0, y_0) = 1.00000$
$x_1 = 0.2$	$y_1 = 0.2027$	$f(x_1, y_1) = 1.04109$
$x_2 = 0.4$	$y_2 = 0.4228$	$f(x_2, y_2) = 1.17876$
$x_3 = 0.6$	$y_3 = 0.6841$	$f(x_3, y_3) = 1.46799$

Example 2

Solution ⇒ Cont...

From the 4th-order Adams-Bashforth, we have

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{24}(55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) \\
&\quad -9f(t_{i-3}, y_{i-3})), \\
y_4 &= y_3 + \frac{0.2}{24}(55f(x_3, y_3) - 59f(x_2, y_2) + 37f(x_1, y_1) \\
&\quad -9f(x_0, y_0)), \\
y_4 &= 0.6841 + \frac{0.2}{24}(55(1.46799) - 59(1.17876) + 37(1.04109) \\
&\quad -9(1.00000) \\
&=
\end{aligned}
$$

From the 4th-order Adams-Bashforth, we have

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{24}(55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) \\
&\quad -9f(x_{i-3}, y_{i-3})), \\
y_5 &= y_4 + \frac{0.2}{24}(55f(x_4, y_4) - 59f(x_3, y_3) + 37f(x_2, y_2) \\
&\quad -9f(x_1, y_1)), \\
y_5 &=
\end{aligned}
$$

Comparison of multistep and RK fourth order method

- The advantage of multistep over single-step RK methods of the same accuracy is that the multistep methods require only one function evaluation per step, while, RK fourth order method requires 4.
- RK methods have the advantage of including being self-starting.

Predictor-Corrector Methods

What is a predictor-corrector method?

- A predictor-corrector method is an algorithm that proceeds in two steps.
- First, the **prediction step** calculates a rough approximation of the desired quantity.
- Second, the **corrector step** refines the initial approximation using another means.

What is a predictor-corrector method?
Cont...

- In the predictor-corrector methods, four prior values are needed for finding the value of y at t_i.
- These methods though slightly complex, have the advantage of giving an estimate of error from successive approximations to y_i.

What is a predictor-corrector method?
Example

If t_i and t_{i+1} be two connective mesh points, such that $t_{i+1} = t_i + h$, then in Euler's method we have

$$y_{i+1} = y_i + hf(t_0 + ih, y_i), \; i = 0, 1, 2, 3, ... \tag{10}$$

The modified Euler's method gives us

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1})\right] \tag{11}$$

What is a predictor-corrector method?
Example ⇒ Cont...

- The value of y_{i+1} is first estimated by (10) and then used in right hand side of (11) giving a better approximation of y_{i+1}.
- This value of y_{i+1} is again substituted in (11) to find a still better approximation of y_{i+1}.
- This procedure is repeated till two consecutive iterated values of y_{i+1} agree.

What is a predictor-corrector method?
Example ⇒ Cont...

- This technique of refining an initially crude estimate of y_{i+1} by means of a more accurate formula is known as predictor-corrector method.
- The equation (10) is therefore called the **predictor** while (11) serves as a **corrector** of y_{i+1}.
- In this section we describe two such methods, namely, Adams-Moulton method and Milne's method.

Adams-Moulton Method

About Adams-Moulton Method

- Here we use Adams-bashforth and Adams-moulton methods as a pair to contruct a predictor-corrector method.
- Also, by using four-step Adams-bashforth and Adams-moulton methods together, the predictor-corrector formula is:

$$
\begin{aligned}
y_{i+1}^{(p)} &= y_i + \frac{h}{24}(55f(t_i, y_i) - 59f(t_{i-1}, y_{i-1}) + 37f(t_{i-2}, y_{i-2}) \\
&\quad -9f(t_{i-3}, y_{i-3})), \\
y_{i+1}^{(c)} &= y_i + \frac{h}{24}(9f(t_{i+1}, y_{i+1}^{(p)}) + 19f(t_i, y_i) - 5f(t_{i-1}, y_{i-1}) \\
&\quad +f(t_{i-2}, y_{i-2})).
\end{aligned}
$$

About Adams-Moulton Method

Cont...

- Note, the four-step Adams-bashforth method needs four initial values to start the calculation.
- It needs to use other methods, for example Runge-Kutta, to get these initial values.

Example 1

Use Adams-Moulton method to find $y(1.4)$ from the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2(1+y),$$

given that $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.543$, $y(1.3) = 1.979$.

Here $f(x, y) = x^2(1 + y)$ and $h = 0.1$.

x_i	y_i	$f(x_i, y_i) = x_i^2(1 + y_i)$
$x_0 = 1.0$	$y_0 = 1.000$	$f(x_0, y_0) = 2.000$
$x_1 = 1.1$	$y_1 = 1.233$	$f(x_1, y_1) = 2.702$
$x_2 = 1.2$	$y_2 = 1.543$	$f(x_2, y_2) = 3.662$
$x_3 = 1.3$	$y_3 = 1.979$	$f(x_3, y_3) = 5.035$

By Adams-Bashforth predictor method, we obtain

$$
\begin{aligned}
y_{i+1}^{(p)} &= y_i + \frac{h}{24}(55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) \\
&\quad -9f(x_{i-3}, y_{i-3})) \\
y_4^{(p)} &= y_3 + \frac{h}{24}(55f(x_3, y_3) - 59f(x_2, y_2) + 37f(x_1, y_1) \\
&\quad -9f(x_0, y_0)) \\
&= 1.979 + \frac{0.1}{24}(55(5.035) - 59(3.662) + 37(2.702) - 9(2.000)) \\
&= 2.573
\end{aligned}
$$

Example 1

Solution ⇒ Cont...

x_i	y_i	$f(x_i, y_i) = x_i^2(1 + y_i)$
$x_0 = 1.0$	$y_0 = 1.000$	$f(x_0, y_0) = 2.000$
$x_1 = 1.1$	$y_1 = 1.233$	$f(x_1, y_1) = 2.702$
$x_2 = 1.2$	$y_2 = 1.543$	$f(x_2, y_2) = 3.669$
$x_3 = 1.3$	$y_3 = 1.979$	$f(x_3, y_3) = 5.035$
$x_4 = 1.4$	$y_4^{(p)} = 2.573$	$f(x_4, y_4^{(p)}) = 7.003$

Using Adams-Moulton corrector method, we obtain

$$\begin{aligned}
y_{i+1}^{(c)} &= y_i + \frac{h}{24}(9f(x_{i+1}, y_{i+1}^{(p)}) + 19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) \\
&\quad + f(x_{i-2}, y_{i-2})) \\
y_4^{(c)} &= y_3 + \frac{h}{24}(9f(x_4, y_4^{(p)}) + 19f(x_3, y_3) - 5f(x_2, y_2) \\
&\quad + f(x_1, y_1)) \\
&= 1.979 + \frac{0.1}{24}(9(7.003) + 19(5.035) - 5(3.669) + 2.702) \\
&= 2.575
\end{aligned}$$

Hence $y(1.4) = 2.575$.

Example 2

Use Adams-Moulton method with $h = 0.2$ to find $y(0.8)$ given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2,\ y(0) = 0.$$

We take $h = 0.2$ with $x_0 = 0$, $y_0 = 0$ and use Runge-Kutta fourth order method to obtain

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = 0.2 \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = 0.202 \\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = 0.20204 \\
k_4 &= hf(x_1, y_0 + k_3) = 0.20816 \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.2027
\end{aligned}$$

By following the same procedure, we can obtain y_2, and y_3.

Thus we have

x_i	y_i	$f(x_i, y_i) = 1 + y_i^2$
$x_0 = 0.0$	$y_0 = 0.0000$	$f(x_0, y_0) = 1.00000$
$x_1 = 0.2$	$y_1 = 0.2027$	$f(x_1, y_1) = 1.04109$
$x_2 = 0.4$	$y_2 = 0.4228$	$f(x_2, y_2) = 1.17876$
$x_3 = 0.6$	$y_3 = 0.6841$	$f(x_3, y_3) = 1.46799$

Example 2
Solution ⇒ Cont...

By Adams-Bashforth predictor method, we obtain

$$
\begin{aligned}
y_{i+1}^{(p)} &= y_i + \frac{h}{24}(55f(x_i, y_i) - 59f(x_{i-1}, y_{i-1}) + 37f(x_{i-2}, y_{i-2}) \\
&\quad -9f(x_{i-3}, y_{i-3})) \\
y_4^{(p)} &= y_3 + \frac{h}{24}(55f(x_3, y_3) - 59f(x_2, y_2) + 37f(x_1, y_1) \\
&\quad -9f(x_0, y_0)) \\
&= 0.6841 + \frac{0.2}{24}(55(1.46799) - 59(1.17876) + 37(1.04109) \\
&\quad -9(1.0000)) \\
&= 1.02337
\end{aligned}
$$

Example 2
Solution ⇒ Cont...

x_i	y_i	$f(x_i, y_i) = 1 + y_i^2$
$x_0 = 0.0$	$y_0 = 0.0000$	$f(x_0, y_0) = 1.00000$
$x_1 = 0.2$	$y_1 = 0.2027$	$f(x_1, y_1) = 1.04109$
$x_2 = 0.4$	$y_2 = 0.4228$	$f(x_2, y_2) = 1.17876$
$x_3 = 0.6$	$y_3 = 0.6841$	$f(x_3, y_3) = 1.46799$
$x_4 = 0.8$	$y_4 = 1.02337$	$f(x_4, y_4^{(p)}) = 2.04729$

Using Adams-Moulton corrector method, we obtain

$$
\begin{aligned}
y_{i+1}^{(c)} &= y_i + \frac{h}{24}(9f(x_{i+1}, y_{i+1}^{(p)}) + 19f(x_i, y_i) - 5f(x_{i-1}, y_{i-1}) \\
&\quad + f(x_{i-2}, y_{i-2})) \\
y_4^{(c)} &= y_3 + \frac{h}{24}(9f(x_4, y_4^{(p)}) + 19f(x_3, y_3) - 5f(x_2, y_2) \\
&\quad + f(x_1, y_1)) \\
&= 0.6841 + \frac{0.2}{24}(9(2.04729) + 19(1.46799) - 5(1.17876) \\
&\quad + (1.04109)) \\
&= 1.0296
\end{aligned}
$$

Milne's Method

About Milne's method

- Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y), y(a) = \alpha. \tag{12}$$

- In general, Milne's method uses following predictor and corrector formulae to approximate solutions of (12).

$$\begin{aligned} y_{i+1}^{(p)} &= y_{i-3} + \frac{4h}{3}\left(2f(t_i, y_i) - f(t_{i-1}, y_{i-1}) + 2f(t_{i-2}, y_{i-2})\right) \\ y_{i+1}^{(c)} &= y_{i-1} + \frac{h}{3}\Big(f(t_{i+1}, y_{i+1}^{(p)}) + 4f(t_i, y_i) + f(t_{i-1}, y_{i-1})\Big) \end{aligned}$$

Example 1

Using the Runge-Kutta method of order 4 to find y for $x = 0.1, 0.2, 0.3$ given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy + y^2,\ y(0) = 1.$$

Continue the solution at $x = 0.4$ by using Milne's method.

Here $f(x, y) = xy + y^2$, $x_0 = 0$, $y_0 = 0$, $h = 0.1$ and by using Runge-Kutta fourth order method to obtain

$$\begin{aligned}
k_1 &= hf(x_0, y_0) = 0.1000 \\
k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = 0.1155 \\
k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = 0.1172 \\
k_4 &= hf(x_1, y_0 + k_3) = 0.13598 \\
y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1169
\end{aligned}$$

By again applying Runge-Kutta method, we can obtain y_2 and y_3 in the same way.

Thus we have starting values for Milne's method as follows:

x_i	y_i	$f(x_i, y_i) = x_i y_i + y_i^2$
$x_0 = 0.0$	$y_0 = 1.00000$	$f(x_0, y_0) = 1.0000$
$x_1 = 0.1$	$y_1 = 1.01169$	$f(x_1, y_1) = 1.3591$
$x_2 = 0.2$	$y_2 = 1.27730$	$f(x_2, y_2) = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f(x_3, y_3) = 2.7132$

Using predictor equation, we have

$$
\begin{aligned}
y_{i+1}^{(p)} &= y_{i-3} + \frac{4h}{3}\left(2f(x_i, y_i) - f(x_{i-1}, y_{i-1}) + 2f(x_{i-2}, y_{i-2})\right) \\
y_4^{(p)} &= y_0 + \frac{4h}{3}\left(2f(x_3, y_3) - f(x_2, y_2) + 2f(x_1, y_1)\right) \\
&= 1.8344
\end{aligned}
$$

$$x_4 = 0.4, y_4^{(p)} = 1.8344 \quad \Rightarrow \quad f(x_4, y_4^{(p)}) = 4.0988$$

Now using corrector

$$\begin{aligned}
y_{i+1}^{(c)} &= y_{i-1} + \frac{h}{3}\Big(f(x_{i+1}, y_{i+1}^{(p)}) + 4f(x_i, y_i) + f(x_{i-1}, y_{i-1})\Big) \\
y_4^{(c)} &= y_2 + \frac{h}{3}\Big(f(x_4, y_4^{(p)}) + 4f(x_3, y_3) + f(x_2, y_2)\Big) \\
&= 1.8386
\end{aligned}$$

Example 1
Solution ⇒ Cont...

$$x_4 = 0.4, y_4^{(c)} = 1.8386 \quad \Rightarrow \quad f(x_4, y_4^{(c)}) = 4.1159$$

Again using the corrector we get

$$\begin{aligned}
y_{i+1}^{(c)} &= y_{i-1} + \frac{h}{3}\Big(f(x_{i+1}, y_{i+1}^{(c)}) + 4f(x_i, y_i) + f(x_{i-1}, y_{i-1})\Big) \\
y_4^{(c)} &= y_2 + \frac{h}{3}\Big(f(x_4, y_4^{(c)}) + 4f(x_3, y_3) + f(x_2, y_2)\Big) \\
y_4 &= 1.2773 + \frac{0.1}{3}(1.8869 + 4(2.7132) + 4.1182) \\
&= 1.8392
\end{aligned}$$

$$x_4 = 0.4, y_4^{(c)} = 1.8392 \quad \Rightarrow \quad f(x_4, y_4^{(c)}) = 4.1182$$

Again using the corrector we get

$$\begin{aligned}
y_{i+1}^{(c)} &= y_{i-1} + \frac{h}{3}\left(f(x_{i+1}, y_{i+1}^{(c)}) + 4f(x_i, y_i) + f(x_{i-1}, y_{i-1})\right) \\
y_4^{(c)} &= y_2 + \frac{h}{3}\left(f(x_4, y_4^{(c)}) + 4f(x_3, y_3) + f(x_2, y_2)\right) \\
&= 1.2773 + \frac{0.1}{3}(1.8869 + 4(2.7132) + 4.1182) \\
&= 1.8392
\end{aligned}$$

Hence Milne's method approximation is 1.8392.

Example 2

Use Milne's method to find $y(0.8)$ and $y(1.0)$ from

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2,\ y(0) = 0,$$

given that $y(0.2) = 0.2027$, $y(0.4) = 0.4228$ and $y(0.6) = 0.6841$.

Example 2

Solution

Here $f(x, y) = 1 + y^2$, $x_0 = 0$, $y_0 = 0$, and $h = 0.2$.
The starting values for Milne's method are

x_i	y_i	$f(x_i, y_i) = 1 + y_i^2$
$x_0 = 0.0$	$y_0 = 0.0000$	$f(x_0, y_0) = 1.0000$
$x_1 = 0.2$	$y_1 = 0.2027$	$f(x_1, y_1) = 1.0411$
$x_2 = 0.4$	$y_2 = 0.4228$	$f(x_2, y_2) = 1.1787$
$x_3 = 0.6$	$y_3 = 0.6841$	$f(x_3, y_3) = 1.4681$

Using Milne's predictor formula, we get

$$\begin{aligned}
y_{i+1}^{(p)} &= y_{i-3} + \frac{4h}{3}\left(2f(x_i, y_i) - f(x_{i-1}, y_{i-1}) + 2f(x_{i-2}, y_{i-2})\right) \\
y_4^{(p)} &= y_0 + \frac{4h}{3}\left(2f(x_3, y_3) - f(x_2, y_2) + 2f(x_1, y_1)\right) \\
&= 0 + \frac{0.8}{3}\left(2(1.4681) - 1.1787 + 2(1.0411)\right) \\
&= 1.0239
\end{aligned}$$

Example 2
Solution ⇒ Cont...

$$x_4 = 0.8, y_4^{(p)} = 1.0239 \quad \Rightarrow \quad f(x_4, y_4^{(p)}) = 2.0480$$

Now the Milne's corrector formula provides us

$$\begin{aligned}
y_{i+1}^{(c)} &= y_{i-1} + \frac{h}{3}\Big(f(x_{i+1}, y_{i+1}^{(p)}) + 4f(x_i, y_i) + f(x_{i-1}, y_{i-1})\Big) \\
y_4^{(c)} &= y_2 + \frac{h}{3}\Big(f(x_4, y_4^{(p)}) + 4f(x_3, y_3) + f(x_2, y_2)\Big) \\
&= 0.4228 + \frac{0.2}{3}(2.0480 + 4(1.4681) + 1.1787) \\
&= 1.0294
\end{aligned}$$

Proceeding on similar lines, we obtain $y_5 = y(1.0) = 1.5549$.

Thank you !