Mathematical Modelling-II

(AMT221β/IMT221β)

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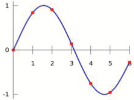
Mathematical Modelling through Difference Equations

Finite Differences

Introduction

- In many problems of real world, an explicit form of a function is not given.
- However, the values of dependent variable for changes in the independent variable are known.
- The change in the independent variable is not continuous but by finite jumps, whether equal or unequal.
- The behavior of the function can be studied with the help of these observations.
- In this chapter, we consider the formulae expresses in terms of differences of the functional values.



Forward difference
The first forward difference

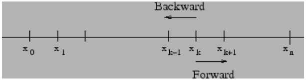
- We define the first forward difference operators, denoted by Δ, as

$$\Delta f(x) = f(x+h) - f(x).$$

- The expression $f(x+h) - f(x)$ gives the first forward difference of $y = f(x)$ and the operator Δ is called the first forward difference operators.

- Given the step size h, this formula uses the values at x and $x + h$, the point at the next step.

- As it is moving in the forward direction, it is called the forward difference operator.



In particular, for $x = x_0$, we get,

$$\begin{aligned} \Delta f(x_0) &= f(x_0 + h) - f(x_0) \\ \Delta y_0 &= y_1 - y_0. \end{aligned}$$

In particular, for $x = x_k$, we get,

$$\begin{aligned} \Delta f(x_k) &= f(x_k + h) - f(x_k) \\ \Delta y_k &= y_{k+1} - y_k. \end{aligned}$$

The second forward difference operator, Δ^2, is defined as

$$\begin{aligned}
\Delta^2 f(x) &= \Delta(\Delta f(x)) \\
&= \Delta(f(x+h) - f(x)) \\
&= \Delta f(x+h) - \Delta f(x) \\
&= f(x+2h) - f(x+h) - f(x+h) + f(x) \\
&= f(x+2h) - 2f(x+h) + f(x)
\end{aligned}$$

In particular, for $x = x_k$, we get,

$$\begin{aligned}
\Delta^2 y_k &= \Delta(\Delta y_k) \\
\Delta^2 y_k &= \Delta(y_{k+1} - y_k) \\
\Delta^2 y_k &= \Delta y_{k+1} - \Delta y_k \\
\Delta^2 y_k &= y_{k+2} - 2y_{k+1} + y_k.
\end{aligned}$$

The r^{th} forward difference operator, Δ^r, is defined as

$$\begin{aligned} \Delta^r f(x) &= \Delta^{r-1} f(x+h) - \Delta^{r-1} f(x), \quad r = 1, 2, 3, ... \\ \text{with } \Delta^0 f(x) &= f(x). \end{aligned}$$

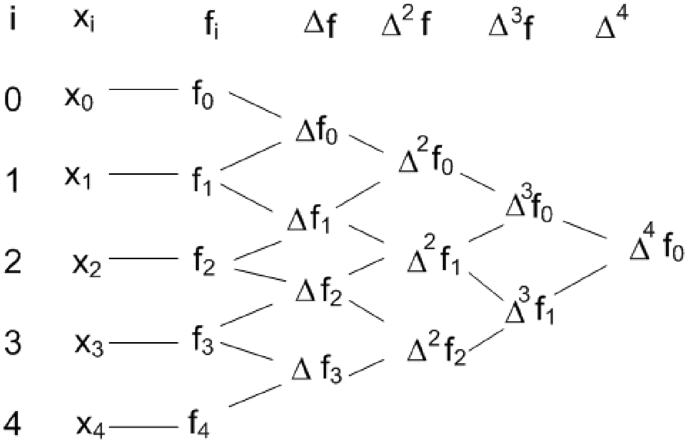
Forward difference

Forward difference table for the function $y = f(x)$

x	**f(x)**			
x_0	$f(x_0)$			
		$\Delta f(x_0)$		
			$\Delta^2 f(x_0)$	
$x_0 + h$	$f(x_0 + h)$			
		$\Delta f(x_0 + h)$		$\Delta^3 f(x_0)$
			$\Delta^2 f(x_0 + h)$	
$x_0 + 2h$	$f(x_0 + 2h)$			
		$\Delta f(x_0 + 2h)$		$\Delta^3 f(x_0 + h)$
			$\Delta^2 f(x_0 + 2h)$	
$x_0 + 3h$	$f(x_0 + 3h)$			
		$\Delta f(x_0 + 3h)$		
$x_0 + 4h$	$f(x_0 + 4h)$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Forward difference

An alternative forward difference table for the function $y = f(x)$



Example 1

Find Δy_4, $\Delta^2 y_2$ and $\Delta^3 y_2$ for the tabulated values of $y = f(x)$.

i	0	1	2	3	4	5
x_i	0	0.1	0.2	0.3	0.4	0.5
y_i	0.03	0.13	0.22	0.38	0.44	0.79

Example 1
Solution

$$
\begin{aligned}
\Delta y_4 &= y_5 - y_4 \\
&= 0.79 - 0.44 \\
&= 0.35 \\
\Delta^2 y_2 &= \Delta(\Delta y_2) \\
&= \Delta(y_3 - y_2) \\
&= \Delta y_3 - \Delta y_2 \\
&= (y_4 - y_3) - (y_3 - y_2) \\
&= y_4 - 2y_3 + y_2 \\
&= 0.44 - 2 \times 0.38 + 0.22 \\
&= -0.1
\end{aligned}
$$

Example 1

Solution⇒Cont...

$$
\begin{aligned}
\Delta^3 y_2 &= \Delta(\Delta^2 y_2) \\
&= \Delta(y_4 - 2y_3 + y_2) \\
&= \Delta y_4 - 2\Delta y_3 + \Delta y_2 \\
&= (y_5 - y_4) - 2(y_4 - y_3) + (y_3 - y_2) \\
&= y_5 - 3y_4 + 3y_3 - y_2
\end{aligned}
$$

Example 2

Show that

$$\Delta \tan^{-1} x = \tan^{-1} \frac{1}{1 + x + x^2},$$

if unity is the interval of differencing.

Example 2
Solution

Let $f(x) = \tan^{-1} x$. Then by applying first forward difference formula, we have

$$
\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta \tan^{-1} x &= \tan^{-1}(x+h) - \tan^{-1} x \\
\Delta \tan^{-1} x &= \tan^{-1}(x+1) - \tan^{-1} x \\
\Delta \tan^{-1} x &= \tan^{-1} \frac{x+1-x}{1+x(x+1)} \\
\Delta \tan^{-1} x &= \tan^{-1} \frac{1}{1+x+x^2}
\end{aligned}
$$

Example 3

Evaluate

$$\Delta x(x+1)(x-2)$$

if unity is the interval of differencing.

Let $f(x) = x(x+1)(x-2)$. Then by applying first forward difference formula, we have

$$\begin{aligned}
\Delta f(x) &= f(x+h) - f(x) \\
\Delta x(x+1)(x-2) &= (x+h)(x+1+h)(x-2+h) - x(x+1)(x-2 \\
\Delta x(x+1)(x-2) &= (x+1)(x+2)(x-1) - x(x+1)(x-2) \\
\Delta x(x+1)(x-2) &= (x+1)(3x-2)
\end{aligned}$$

Remark

- If f is a constant ⇨ $\Delta f = 0$.
- If k is a constant ⇨ $\Delta(kf) = k(\Delta f)$.

Exercise

If $\Delta^3(1 - ax)(1 - 3x)(1 - 4x) = 72$, find a, given unity as the interval of differencing.

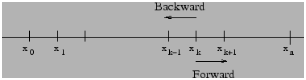
Answer is $a = -1$

- The first backward difference operator denoted by ∇, is defined as

$$\nabla f(x) = f(x) - f(x-h).$$

- Given the step size h, note that this formula uses the values at x and $x-h$, the point at the previous step.

- As it moves in the backward direction, it is called the backward difference operator.



- The second backward difference operator denoted by ∇^2, is defined as

$$\begin{aligned}
\nabla^2 f(x) &= \nabla(\nabla f(x)) \\
&= \nabla\left(f(x) - f(x-h)\right) \\
&= \nabla f(x) - \nabla f(x-h) \\
&= f(x) - f(x-h) - f(x-h) + f(x-2h) \\
&= f(x) - 2f(x-h) + f(x-2h)
\end{aligned}$$

The r^{th} backward difference operator, ∇^r, is defined as

$$\begin{aligned} \nabla^r f(x) &= \nabla^{r-1} f(x) - \nabla^{r-1} f(x-h), \quad r = 1, 2, 3.. \\ \text{with } \nabla^0 f(x) &= f(x). \end{aligned}$$

In particular, for $x = x_k$, we get

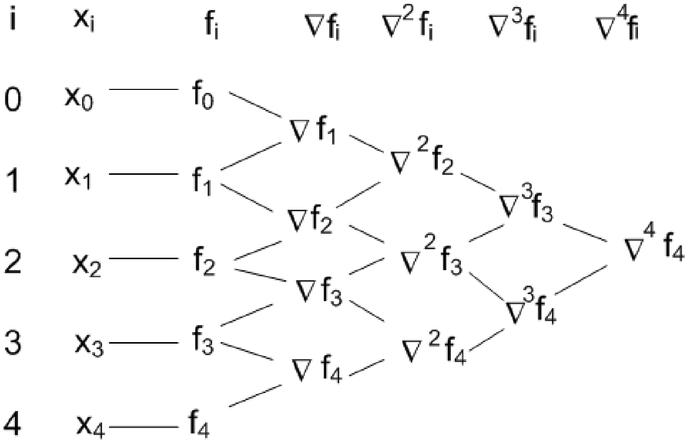
$$\nabla y_k = y_k - y_{k-1}$$

and

$$\begin{aligned}\nabla^2 y_k &= \nabla y_k - \nabla y_{k-1} \\ &= y_k - 2y_{k-1} + y_{k-2}\end{aligned}$$

Backward difference

Backward difference table for the function $y = f(x)$



Remark

In general it can be shown that $\Delta^k f(x) = \nabla^k f(x + kh)$ or $\Delta^k y_m = \nabla^k y_{k+m}$.

Example

Find ∇y_5, $\nabla^2 y_2$ and $\nabla^3 y_3$ for the tabulated values of $y = f(x)$.

i	0	1	2	3	4	5
x_i	0	0.1	0.2	0.3	0.4	0.5
y_i	0.03	0.13	0.22	0.38	0.44	0.79

Example
Solution

$$
\begin{aligned}
\nabla y_5 &= y_5 - y_4 \\
&= 0.79 - 0.44 \\
&= 0.35 \\
\nabla^2 y_2 &= \nabla(y_2 - y_1) \\
&= \nabla y_2 - \nabla y_1 \\
&= y_2 - y_1 - y_1 + y_0 \\
&= y_2 - 2y_1 + y_0 \\
&= 0.22 - 2 \times 0.13 + 0.03
\end{aligned}
$$

Example
Solution⇒Cont...

$$
\begin{aligned}
\nabla^3 y_3 &= \nabla^2(\nabla y_3) \\
&= \nabla^2(y_3 - y_2) \\
&= \nabla^2 y_3 - \nabla^2 y_2 \\
&= \nabla(y_3 - y_2) - \nabla(y_2 - y_1) \\
&= (y_3 - y_2 - y_2 + y_1) - (y_2 - y_1 - y_1 + y_0) \\
&= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \\
&= y_3 - 3y_2 + 3y_1 - y_0
\end{aligned}
$$

The first central difference operator, denoted by δ, is defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right).$$

Central difference
The second central difference

$$\delta^2 f(x) = ?$$

Central difference

The second central difference

$$
\begin{aligned}
\delta^2 f(x) &= \delta f\left(x+\frac{h}{2}\right)-\delta f\left(x-\frac{h}{2}\right) \\
&= f\left(x+\frac{h}{2}+\frac{h}{2}\right)-f\left(x+\frac{h}{2}-\frac{h}{2}\right) \\
&\quad -f\left(x-\frac{h}{2}+\frac{h}{2}\right)+f\left(x-\frac{h}{2}-\frac{h}{2}\right) \\
&= f(x+h)-2f(x)+f(x-h)
\end{aligned}
$$

Central difference

The r^{th} central difference

$$\begin{aligned} \delta^r f(x) &= \delta^{r-1} f\left(x + \frac{h}{2}\right) - \delta^{r-1} f\left(x - \frac{h}{2}\right) \\ \text{with } \delta^0 f(x) &= f(x). \end{aligned}$$

In particular, for $x = x_k$, define $y_{k+\frac{1}{2}} = f(x_k + \frac{h}{2})$, and $y_{k-\frac{1}{2}} = f(x_k - \frac{h}{2})$, then

$$\begin{aligned} \delta f_k &= f_{k+\frac{1}{2}} - f_{k-\frac{1}{2}} \\ \delta y_k &= y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}}, \end{aligned}$$

and

$$\delta^2 y_k = y_{k+1} - 2y_k + y_{k-1}.$$

Central difference

Central difference table for the function $y = f(x)$

x	f	δf	δ^2f	δ^3f	δ^4f
x_0	f_0				
		$\delta f_{1/2}$			
			$\delta^2 f_1$		
x_1	f_1				
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
			$\delta^2 f_2$		$\delta^4 f_2$
x_2	f_2				
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
			$\delta^2 f_3$		
x_3	f_3				
		$\delta f_{7/2}$			
x_4	f_4				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Shift operator

The shift operator E is defined as

$$Ef(x) = f(x+h), \text{ where } h > 0.$$

And also

$$Ef_i = f_{i+1}.$$

The power of E are defined by

$$\begin{aligned} E^r f(x) &= f(x+rh), \quad r = 1, 2, \ldots \\ E^0 f(x) &= f(x). \end{aligned}$$

The relationship between E and Δ can be obtained as follows:

$$\begin{aligned}\Delta f(x) &= f(x+h) - f(x)\\ &= Ef(x) - f(x) \text{ or}\\ \Delta f(x) &= (E-1)f(x).\end{aligned}$$

Using this relation, we can express $\Delta^r f(x)$ in terms of the values of $f(x)$. That is, we treat $(E-1)$ as an operator which can be manipulated by the rules of algebra, provided it precedes an entity on which it operates. Therefore

$$\Delta^r f(x) = (E-1)^r f(x).$$

The relationship between operators is given by

$$
\begin{aligned}
\delta f(x) &= f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right) \\
&= E^{1/2}f(x)-E^{-1/2}f(x) \\
&= (E^{1/2}-E^{-1/2})f(x), \text{ so that} \\
\delta &= E^{1/2}-E^{-1/2}.
\end{aligned}
$$

Inverse operator

The inverse operator E^{-1} is defined as

$$E^{-1}f(x) = f(x-h), \text{ where } h > 0.$$

Inverse operator
Powers of inverse operator

$$
\begin{aligned}
E^{-2}f(x) &= f(x-2h)\\
E^{-3}f(x) &= f(x-3h)\\
&\vdots\\
E^{-r}f(x) &= f(x-rh)\\
E^{-1/2}f(x) &= f\left(x-\frac{1}{2}h\right)\\
E^{-1/2}f_i &= f_{i-\frac{1}{2}}
\end{aligned}
$$

Averaging operator

The averaging operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right].$$

The relationship between operators is given by

$$\begin{aligned}\mu f(x) &= \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\\ &= \frac{1}{2}\left[E^{1/2}f(x)+E^{-1/2}f(x)\right]\\ &= \frac{1}{2}(E^{1/2}+E^{-1/2})f(x), \text{ so that}\\ \mu &= \frac{1}{2}\left[E^{1/2}+E^{-1/2}\right].\end{aligned}$$

Differential operator

Differential operators are a generalization of the operation of differentiation. The simplest differential operator D, acting on a function $f(x)$, returns the first derivative of this function:

$$Df(x) = f'(x).$$

Double D allows to obtain the second derivative of the function:

$$D^2 f(x) = f''(x).$$

Similarly, the r^{th} power of D leads to the r^{th} derivative:

$$D^r f(x) = f^{(r)}(x).$$

Divided differences

The first divided difference of $f(x)$ between x_0 and x_1 is denoted by $f[x_0, x_1]$ and is defined as

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$

and similarly, between the arguments x_1 and x_2 as

$$f[x_1, x_2] = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Note: In this course unit square brackets are used for divided differences.

The second-order divided difference between three arguments x_0, x_1 and x_2 denoted by $f[x_0, x_1, x_2]$ is defined as

$$f[x_0, x_1, x_2] = \frac{f[x_0, x_1] - f[x_1, x_2]}{x_0 - x_2}.$$

In general, $(r+1)^{\text{th}}$ divided difference denoted by $f[x_0, x_1, x_2, ..., x_{r+1}]$ is defined as

$$f[x_0, x_1, x_2, ..., x_{r+1}] = \frac{f[x_0, x_1, x_2, ..., x_r] - f[x_1, x_2, x_3, ..., x_{r+1}]}{x_0 - x_{r+1}}.$$

Some important relations between the operators

1. $\nabla = 1 - E^{-1}$
2. $\Delta = E\nabla$
3. $\Delta - \nabla = \Delta\nabla$
4. $\mu^2 = 1 + \frac{1}{4}\delta^2$
5. $\delta^2 = \Delta - \nabla$

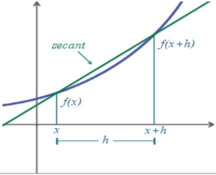
Why do we need differences?
Numerical differentiation

- The simplest way to compute a function's derivatives numerically is to use finite difference approximations.
- Suppose we are interested in computing the first derivative of a smooth function $f : \mathbb{R} \to \mathbb{R}$.
- Then from the definition of a derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- If we choose small h, it can be approximated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$



- This is the easiest and most intuitive finite difference formula and it is called the forward difference.
- The forward difference is the most widely used way to compute numerical derivatives but often it is not the best choice.

Why do we need differences?
Numerical differentiation⇒Cont...

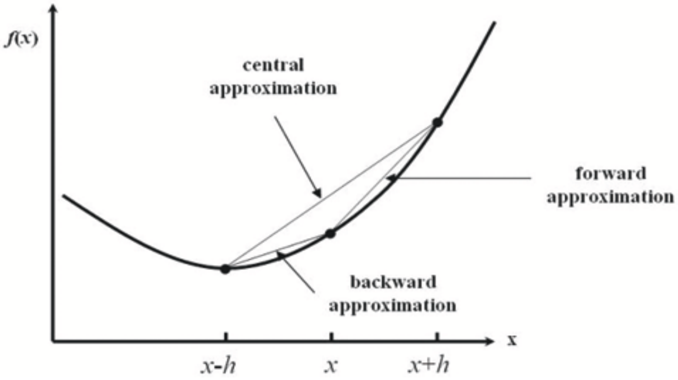


Figure: Differences for first derivative

Error bound for the forward difference

In order to compare to alternative approximations we need to derive an error bound for the forward difference. This can be done by taking a Taylor expansion of $f(x+h)$,

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x)... \\
f(x+h) - f(x) &= hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x)... \\
\frac{f(x+h) - f(x)}{h} &= f'(x) + \frac{h}{2!}f''(x) + \frac{h^2}{3!}f'''(x) + \frac{h^3}{4!}f^{iv}(x)... \\
\frac{f(x+h) - f(x)}{h} &= f'(x) + O(h) \\
f'(x) &\approx \frac{f(x+h) - f(x)}{h}.
\end{aligned}
$$

Error bound for the forward difference
Cont...

- We say that this approximation is first-order accurate since the dominate term in the truncation error is $O(h)$.
- This means that the error of the forward difference approximation of the first derivative is proportional to the step size h.

Error bound for the backward difference

Taking the Taylor series expansion of $f(x-h)$ about x, as:

$$\begin{aligned}
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x)... \\
f(x) - f(x-h) &= hf'(x) - \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) - \frac{h^4}{4!}f^{iv}(x)... \\
\frac{f(x) - f(x-h)}{h} &= f'(x) - \frac{h}{2!}f''(x) + \frac{h^2}{3!}f'''(x) - \frac{h^3}{4!}f^{iv}(x)... \\
\frac{f(x) - f(x-h)}{h} &= f'(x) + O(h) \\
f'(x) &\approx \frac{f(x) - f(x-h)}{h}
\end{aligned}$$

This equation has also an error of $O(h)$.

Error bound for the central difference

In order to derive another approximation formula for the first derivative, we tack the Taylor series expansion of $f(x+h)$ and $f(x-h)$:

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \dots \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) - \dots
\end{aligned}
$$

Error bound for the central difference

Cont...

Divide the difference between these two equations by $2h$ to get the central difference approximation for the first derivative as:

$$\begin{aligned}
\frac{f(x+h)-f(x-h)}{2h} &= f'(x)+\frac{h^2}{3!}f'''(x)+\frac{h^4}{5!}f^{v}(x)+\ldots \\
\frac{f(x+h)-f(x-h)}{2h} &= f'(x)+O(h^2) \\
f'(x) &\approx \frac{f(x+h)-f(x-h)}{2h}.
\end{aligned}$$

Which can be regarded as an improvement over forward and backward, since it has the truncation error of $O(h^2)$ for $|h| < 1$.

Remark

- Notice that the central difference approximation is second-order accurate since the dominate term in its truncation error is $O(h^2)$.
- Thus the central difference is more accurate than the forward and backward difference due to its smaller truncation error.

Approximation error

Given some value v and its approximation v_{approx}, the **absolute error** is

$$\epsilon = |v - v_{\text{approx}}|,$$

where the vertical bars denote the absolute value. If $v \neq 0$, the **relative error** is

$$\eta = \frac{|v - v_{\text{approx}}|}{|v|},$$

and the **percent error** is

$$\lambda = \frac{|v - v_{\text{approx}}|}{|v|} \times 100.$$

These definitions can be extended to the case when v and v_{approx} are n-dimensional vectors, by replacing the absolute value with an n-norm.

Example

Suppose you are given $f(x) = \ln x$, then

a) calculate the exact value of $f'(1.8)$.

b) obtain forward difference approximation for $f'(1.8)$, if $h = 0.1$.

c) obtain backward difference approximation for $f'(1.8)$, if $h = 0.1$.

d) obtain central difference approximation for $f'(1.8)$, if $h = 0.1$.

e) calculate percent error for each of the above approximation.

f) repeat calculation, if $h = 0.01$.

Example
Solution

a)

$$
\begin{aligned}
f(x) &= \ln x \\
f'(x) &= \frac{1}{x} \\
f'(1.8) &= \frac{1}{1.8} = 0.555555
\end{aligned}
$$

b)

$$
\begin{aligned}
f'(x) &\approx \frac{f(x+h) - f(x)}{h} \\
f'(1.8) &\approx \frac{f(1.9) - f(1.8)}{0.1} \\
&\approx 0.540672
\end{aligned}
$$

c)

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$
$$f'(1.8) \approx \frac{f(1.8) - f(1.7)}{0.1} = 0.571584$$

d)

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$
$$f'(1.8) \approx \frac{f(1.9) - f(1.7)}{0.2} = 0.556128$$

e)

$$\text{Forward difference} \Rightarrow \frac{|0.555555 - 0.540672|}{|0.555555|} \times 100$$
$$= 2.68\%$$
$$\text{Backward difference} \Rightarrow \frac{|0.555555 - 0.571584|}{|0.555555|} \times 100$$
$$= 2.88\%$$
$$\text{Central difference} \Rightarrow \frac{|0.555555 - 0.556128|}{|0.555555|} \times 100$$
$$= 0.10\%$$

f)

$$
\begin{aligned}
\text{Forward difference} &\Rightarrow f'(1.8) \simeq \frac{f(1.81)-f(1.8)}{0.01} = 0.554018 \\
&\Rightarrow \text{percent error} = 0.27\% \\
\text{Backward difference} &\Rightarrow f'(1.8) \simeq \frac{f(1.8)-f(1.79)}{0.01} = 0.557104 \\
&\Rightarrow \text{percent error} = 0.28\% \\
\text{Central difference} &\Rightarrow f'(1.8) \simeq \frac{f(1.81)-f(1.79)}{0.02} = 0.555561 \\
&\Rightarrow \text{percent error} = 0.0001\%
\end{aligned}
$$

Exercise

Suppose you are given $f(x) = \sqrt{x}$, then

a) calculate the exact value of $f'(2)$.

b) obtain forward difference approximation for $f'(2)$, if $h = 0.2$.

c) obtain backward difference approximation for $f'(2)$, if $h = 0.2$.

d) obtain central difference approximation for $f'(2)$, if $h = 0.2$.

e) calculate percent error for each of the above approximation.

f) repeat calculation, if $h = 0.02$.

Exercise
Solution

a) 0.353553
b) 0.345130
c) 0.362864
d) 0.353997
e) 2.38%, 2.63%, 0.12%
f) 0.352674 (0.25%), 0.354442 (0.25%), 0.3535578 (0.0012%)

Approximation for the second derivative

In order to obtain an approximation formula for the second derivative, we take the Taylor series expansion of $f(x+h)$ and $f(x-h)$:

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \dots \\
f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) - \dots
\end{aligned}
$$

Adding these two equations:

$$
f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + 2\frac{h^4}{4!}f^{iv}(x) + \dots
$$

Subtracting $2f(x)$ from both sides and dividing both sides by h^2 yields:

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x)+\frac{h^2}{12}f^{iv}(x)+...$$

Which has a truncation error of $O(h^2)$.

$$f''(x)\approx\frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Remark

- We can find finite difference approximations for other higher order derivatives using a similar approach.

Difference equations

Why do we need difference equations?

- Some real world relations are recorded at fixed time intervals.
- For example, income, expenditure, savings and inflation could be measured daily, weekly, monthly, quarterly, annually and so on.
- Equations that relate such quantities are referred to as **difference equations**.
- **Difference equations** are the discrete time counterparts of **diferential equations** which we will also study in this course.

Differential and difference equations

- **Differential equations** are those in which an equality is expressed in terms of a function of one or more independent variables and **derivatives** of the function with respect to one or more of those independent variables.
- **Difference equations** are those in which an equality is expressed in terms of a function of one or more independent variables and **finite differences** of the function.

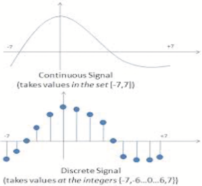
Differential and difference equations

Cont...

- **Differential equations** are important in signal and system analysis because they describe the dynamic behavior of **continuous time** physical systems.
- **Difference equations** are important in signal and system analysis because they describe the dynamic behavior of **discrete time** systems.

Differential and difference equations
Cont...

- Discrete time is equally-spaced points in time, separated by some time difference Δt.
- In discrete time signals and systems the behavior of a signal and the action of a system are known only at discrete points in time and are not defined between those discrete points in time.



Connection between difference and differential equations

- Let us consider the first order ordinary differential equation,

$$3\frac{dy(t)}{dt} + 5y(t) = 0. \tag{1}$$

- It can be approximated by a difference equation.
- We can do this by approximating derivatives by finite differences.

Connection between difference and differential equations
Cont...

Recall these definitions of a derivative,

$$
\begin{aligned}
\frac{dy(t)}{dt} &= \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}, \\
\frac{dy(t)}{dt} &= \lim_{\Delta t \to 0} \frac{y(t) - y(t-\Delta t)}{\Delta t}, \\
\frac{dy(t)}{dt} &= \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t-\Delta t)}{2\Delta t}.
\end{aligned}
$$

- At any point at which $y(t)$ is differentiable, any of these definitions of a derivative yield exactly the same result when the limit is taken.

- A derivative in continuous time can be approximated by a finite difference in discrete time by

$$\frac{y\left((n+1)\Delta t\right) - y\left(n\Delta t\right)}{\Delta t} \quad \Leftarrow \quad \text{Forward difference}$$

$$\frac{y\left(n\Delta t\right) - y\left((n-1)\Delta t\right)}{\Delta t} \quad \Leftarrow \quad \text{Backward difference}$$

$$\frac{y\left((n+1)\Delta t\right) - y\left((n-1)\Delta t\right)}{2\Delta t} \quad \Leftarrow \quad \text{Central difference}$$

Connection between difference and differential equations
Cont...

- As an illustration we will convert the differential equation (1), to a difference equation by using a forward difference approximation,

$$3\frac{y\left((n+1)\Delta t\right)-y\left(n\Delta t\right)}{\Delta t}+5y(n\Delta t)=0.$$

- Here $y(n\Delta t)$ is used to represent the n^{th} value of y.
- It can also be denoted as y_n.
- By using that familiar notation, the above equation can be expressed as

$$3(y_{n+1}-y_n)+5\Delta t\ y_n=0.$$

- To solve the above difference equation, we can rewrite it in recursion form

$$y_{n+1} = \frac{3 - 5\Delta t}{3} y_n.$$

- In recursion form, the difference equation expresses the next value of y (say y_{n+1}) in terms of the present value of y (say y_n).

Definition of difference equation

- A **difference equation**, also called **recurrence equation**, is an equation that defines a sequence recursively.
- Each term of the sequence is defined as a function of the previous terms of the sequence:

$$y_{n+1} = f(y_n, y_{n-1}, ...y_0).$$

Example 1

Fibonacci sequence are the numbers in the following integer sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 35, ...$$

Define it using a recurrence equation.

$$y_n = y_{n-1} + y_{n-2} \tag{2}$$

Equation (2) defines the so-called Fibonacci sequence. Starting with $y_0 = 1$ and $y_1 = 1$, we can easily calculate each following terms of the sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 35, ...$$

Note that each term can be computed only if the two first terms of the sequence are given. Those terms are called the initial conditions of the system.

Example 2

Suppose a certain polpulation of rabits is growing at the rate of 3% per year. If we let y_0 represents the size of the initial population of rabits and y_n represents the number of rabits n years, then define population growth using a recurrence equation.

Example 2
Solution

$$\begin{aligned}
y_1 &= y_0 + 0.03y_0 \\
y_2 &= y_1 + 0.03y_1 \\
&. \\
&. \\
&. \\
y_n &= y_{n-1} + 0.03y_{n-1} \\
y_{n+1} &= y_n + 0.03y_n
\end{aligned}$$

Example 3

Radium is a radioactive element which decays at a rate of 1% every 25 years. This means that amount left at the beginning of any given 25 years period is equal to the amount at the beginning of the previous 25 years period minus 1% of that amount. If y_0 is the initial amount of the Radium and y_n is the amount of Radium still remaining after 25 years, then define radioactive decay using a recurrence equation.

Example 3
Solution

$$
\begin{aligned}
y_1 &= 0.99y_0 \\
y_2 &= 0.99y_1 = 0.99(0.99y_0) = (0.99)^2 y_0 \\
y_3 &= 0.99y_2 = 0.99(0.99y_1) = (0.99)^3 y_0 \\
&\vdots \\
y_n &= = 0.99y_{n-1} = (0.99)^n y_0
\end{aligned}
$$

Classification of difference equations

- As with differential equations, one can refer to the order of a difference equation and note whether it is linear or non-linear and whether it is homogeneous or non homogeneous.

Order of difference equations

- The order of a linear recurrence relation is the number of preceeding terms required by the definition.

$$
\begin{aligned}
x_t = -5x_{t-1} + 9 &\quad \Leftarrow \quad \text{First order} \\
x_n = rx_{n-1}(1 - x_{n-1}) &\quad \Leftarrow \quad \text{First order} \\
x_n = x_{n-1} + x_{n-2} &\quad \Leftarrow \quad \text{Second order} \\
x_{n+2} - 2x_{n+1} + 2x_n = 0 &\quad \Leftarrow \quad \text{Second order}
\end{aligned}
$$

Linear difference equations

- A difference equation is said to be linear when each term of the sequence is defined as a linear function of the preceding terms.

$$
\begin{aligned}
x_n = x_{n-1} + x_{n-2} \quad &\Leftarrow \quad \text{Linear} \\
x_n = r x_{n-1}(1 - x_{n-1}) \quad &\Leftarrow \quad \text{Non linear}
\end{aligned}
$$

Homogeneous difference equation

- The general form of a linear recurrence relation of order p is as follows:

$$x_n = a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + ... + a_{n-p}x_{n-p} + a_0.$$

- If the coefficients a_i does not depend on n, then the recurrence relation is said to have constant coefficients.

- In addition, if $a_0 = 0$, the recurrence relation is said to be **homogeneous**.

Solving a difference equation

- Solving a difference equation means to find an explicit relation between x_n and the initial conditions.
- For example the solution of $x_n = x_{n-1} + x_{n-2}$, would allow us to evaluate x_{520} (given x_0 and x_1) without computing all the 520 intermediary values.
- The method to solve a difference equation depends on the type of equation we have.

Solving first order linear difference equations

The simplest first-order difference equation is the homogeneous equation:

$$x_{n+1} = ax_n.$$

The solution can be found easily:

$$x_{n+1} = ax_n = a(ax_{n-1}) = ... = a(a(a...(ax_0))).$$

Generally we have:

$$x_n = a^n x_0.$$

The behavior of the system depends on the value of a.

Example 1

Solve $x_{n+1} = 10x_n$, where $x_0 = 1$, and find x_{25}.

The solution can be found easily:

$$\begin{aligned} x_{n+1} &= 10x_n \\ &= 10(10x_{n-1}) = ... = 10(10(10...(10x_0))) \end{aligned}$$

Generally we have $x_n = (10)^n x_0$. Then

$$\begin{aligned} x_{25} &= (10)^{25} \times 1 \\ x_{25} &= (10)^{25} \end{aligned}$$

Example 2

Solve $v_n = 5v_{n-1}$, where $v_1 = 2$, and find v_8.

Example 2
Solution

$$
\begin{aligned}
v_n &= (5)^{n-1} v_1 \\
&= 2 \times (5)^{n-1} \\
v_8 &= (5)^{8-1} \times 2 \\
&= 2 \times (5)^7
\end{aligned}
$$

Solving first order non-homogeneous difference equation

More generally, non-homogeneous first-order difference equation takes the form:

$$x_{n+1} = ax_n + b.$$

The solution can be found easily:

$$\begin{aligned} x_1 &= ax_0 + b \\ x_2 &= ax_1 + b = a(ax_0 + b) + b = a^2x_0 + (a+1)b \\ x_3 &= ax_2 + b = a(a^2x_0 + (a+1)b) + b = a^3x_0 + (a^2 + a + 1)b \end{aligned}$$

Generally we have:

$$x_n = a^n x_0 + (a^{n-1} + a^{n-2} + ... + a + 1)b$$

Solving first order non-homogeneous difference equation
Cont...

Using the summation formula for a geometric series we have:

$$x_{n+1} = ax_n + b \Leftrightarrow x_n = a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}(a \neq 1)$$

Example 1

Solve $x_{n+1} = x_n + 1$, with the initial condition $x_0 = 1$.

The solution can be found easily:

$$
\begin{aligned}
x_1 &= x_0 + 1 \\
x_2 &= x_1 + 1 = (x_0 + 1) + 1 = x_0 + 2 \\
x_3 &= x_2 + 1 = (x_0 + 2) + 1 = x_0 + 3 \\
&\;. \\
&\;. \\
&\;. \\
x_n &= x_0 + n \\
\text{Since } x_0 &= 1, \text{we have} \\
x_n &= 1 + n.
\end{aligned}
$$

Example 2

Solve $x_{n+1} = -3x_n + 4$, with the initial condition $x_0 = 2$.

Example 2
Solution

$$x_{n+1} = ax_n + b \quad \Leftrightarrow \quad x_n = a^n\left(x_0 - \frac{b}{1-a}\right) + \frac{b}{1-a}(a \neq 1)$$

$$x_{n+1} = -3x_n + 4 \quad \Leftrightarrow \quad x_n = (-3)^n\left(2 - \frac{4}{1-(-3)}\right) + \frac{4}{1-(-3)}$$

$$\Leftrightarrow \quad x_n = (-3)^n + 1.$$

Solving second order linear difference equations

A general second order linear difference equation can be written:

$$x_{n+2} + ax_{n+1} + bx_n = 0.$$

Its solution is:

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n,$$

where λ_1 and λ_2 are the solutions of the characteristic equation:

$$\lambda^2 + a\lambda + b = 0.$$

The behavior of the system depends on the value of λ_1 and λ_2.

Example 1

Solve the following second order linear difference equation with the initial condition $x_0 = 1$ and $x_1 = 1$:

$$x_n = x_{n-1} + x_{n-2}.$$

This equation can be rewritten

$$x_{n+2} - x_{n+1} - x_n = 0. \tag{3}$$

If we try the solution $x_n = \lambda^n$, we obtain

$$\begin{aligned} x_{n+1} &= \lambda^{n+1}, \\ x_{n+2} &= \lambda^{n+2}. \end{aligned}$$

Substituting this into (3), and ignoring the case $\lambda = 0$, we obtain the following characteristic equation:

$$\lambda^2 - \lambda - 1 = 0.$$

The roots of the characteristic equation are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = \phi \text{ (Golden Ratio)}$$
$$\text{and } \lambda_2 = \frac{1-\sqrt{5}}{2} = 1-\phi.$$

Successive terms in the Fibonacci sequence are given by:

$$x_n = c_1\phi^n + c_2(1-\phi)^n.$$

Example 1

Solution⇒Cont...

The constants c_1 and c_2 are then determined by the initial conditions. Thus, if $x_0 = 1$ and $x_1 = 1$, then

$$\begin{aligned} 1 &= c_1\phi^0 + c_2(1-\phi)^0, \\ 1 &= c_1\phi^1 + c_2(1-\phi)^1. \end{aligned}$$

Solving for c_1 and c_2 we get

$$\begin{aligned} c_1 &= \frac{\phi}{2\phi - 1} \text{ and} \\ c_2 &= \frac{\phi - 1}{2\phi - 1}. \end{aligned}$$

Hence

$$x_n = \frac{\phi^{n+1} - (1-\phi)^{n+1}}{2\phi - 1} \text{ where } \phi = \frac{1+\sqrt{5}}{2}.$$

Example 2

Solve the difference equation

$$x_{n+2} - 5x_{n+1} + 6x_n = 0,$$

subject to $x_0 = 2$ and $x_1 = 5$.

Example 2
Solution

If we try the solution $x_n = \lambda^n$ we obtain

$$\begin{aligned} x_{n+1} &= \lambda^{n+1}, \\ x_{n+2} &= \lambda^{n+2}. \end{aligned}$$

Substituting this into (4), and ignoring the case $\lambda = 0$, we obtain the following characteristic equation:

$$\lambda^2 - 5\lambda + 6 = 0.$$

The roots of the characteristic equation are $\lambda_1 = 2$ and $\lambda_2 = 3$.

So we may write

$$x_n = c_1 2^n + c_2 3^n.$$

In order to find the values of c_1 and c_2 we use the initial conditions $x_0 = 2$ and $x_1 = 5$.

Hence, we obtain $2 = c_1 + c_2$ and $5 = 2c_1 + 3c_2$ which yields $c_1 = 1$ and $c_2 = 1$.

So the solution is $x_n = 2^n + 3^n$.

Exercise

Solve $u_{k+2} - 6u_{k+1} + 9u_k = 0$ with $u_0 = 1$ and $u_1 = 4$.

The solution is $u_k = 3^k + \frac{1}{3}k3^k$.

More on solutions of difference equations

- The solution of a difference equation of order n shall generally contain n arbitrary constants.
- A solution involving as many arbitrary constants as is the order of the equation, is called the **general solution**.
- Any solution obtained from the general solution by assigning particular values to the arbitrary constants is called a **particular solution**.

More on solutions of difference equations
Cont...

- Consider the difference equation $x_{n+1} - ax_n = 0$, $a \neq 1$.
- The relation $x_n = x_0 a^n$ is a solution of the above difference equation.
- The relation $x_n = x_0 a^n$ is the **general solution**.
- $x_n = 3a^n$ is a **particular solution**.

Consider the difference equation of the form

$$x_{n+r} + k_1 x_{n+r-1} + ... + k_r x_n = g(n), \tag{4}$$

where $k_1, k_2, ..., k_r$ are constants and $g(n)$ is functions of n or constant, is called linear difference equation with constant coefficients.

Case I ➪ $g(n) = 0 \Rightarrow$ homogeneous

Case II ➪ $g(n) \neq 0 \Rightarrow$ non-homogeneous

The equation (4) in homogeneous form can be rewritten as

$$(E^r + k_1 E^{r-1} + ... + k_r)x_n = 0, \tag{5}$$

where E is the shift operator such that $E^r x_n = x_{n+r}$.

If $f(E) = E^r + k_1 E^{r-1} + ... + k_r$, then $f(E) = 0$ is called the **auxiliary equation** and $f(E)$ is called the **characteristic function** of (5).

(a) If the auxiliary equation has r distinct roots $\alpha_1, \alpha_2, ..., \alpha_r$, then the general solution of (5) is

$$x_n = c_1\alpha_1^n + c_2\alpha_2^n + ... + c_r\alpha_r^n,$$

where $c_1, c_2, ..., c_r$ are arbitrary constants.

(b) If the auxiliary equation has real repeated roots, say α_1 repeated p times, α_2 repeated q times, then the general solution of (5) is

$$x_n = (c_1 + c_2 n + ... + c_p n^{p-1})\alpha_1^n + (b_1 + b_2 n + ... + b_q n^{q-1})\alpha_2^n.$$

(c) If the auxiliary equation has non-repeated complex roots, say two of them be $\alpha_1 = \alpha + i\beta$ and $\alpha_2 = \alpha - i\beta$ then the general solution of (5) is

$$x_n = r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$ and c_1, c_2 are arbitrary constants.

(d) If the auxiliary equation has repeated complex roots, say $\alpha + i\beta$ and $\alpha - i\beta$ both repeated twice then the corresponding two terms of the general solution shall be

$$r^n[(c_1 + c_2 n) \cos n\theta + (c_3 + c_4 n) \sin n\theta].$$

Example 1

Solve $x_{n+2} - x_n = 0$.

Example 1
Solution

The given equation can be written as

$$(E^2 - 1)x_n \quad = \quad 0.$$

The auxiliary equation is $(E^2 - 1) = 0$ or $(E - 1)(E + 1) = 0$.

It has two distinct roots 1 and -1.

Therefore, the general solution is $c_1(-1)^n + c_2(1)^n$.

Example 2

Solve $u_{x+2} - 8u_{x+1} + 15u_x = 0$ by the method of difference.

The given equation can be written as

$$(E^2 - 8E + 15)u_x = 0.$$

The auxiliary equation is $E^2 - 8E + 15 = 0$.

It has two distinct roots 3 and 5.

Thus the general solution is given by $u_x = c_1 3^x + c_2 5^x$, where c_1, c_2 are arbitrary constants.

Example 3

Solve the difference equation

$$16x_{n+2} - 8x_{n+1} + x_n = 0.$$

Example 3
Solution

The given equation can be written as

$$(16E^2 - 8E + 1)x_n = 0.$$

The auxiliary equation is $16E^2 - 8E + 1 = 0$.

It has two equal roots 1/4, 1/4.

Thus the general solution is given by $x_n = (c_1 + c_2 n)\left(\frac{1}{4}\right)^n$, where c_1 ,c_2 are arbitrary constants.

Example 4

Solve the difference equation

$$x_{n+2} - 4x_{n+1} + 13x_n = 0.$$

Example 4
Solution

The given equation can be written as

$$(E^2 - 4E + 13)x_n = 0$$

The auxiliary equation is $E^2 - 4E + 13 = 0$.

It has complex roots $2 + 3i$ and $2 - 3i$.

Thus the general solution is given by

$$\begin{aligned} x_n &= r^n(c_1 \cos n\theta + c_2 \sin n\theta), \text{ where} \\ r &= \sqrt{4+9} = \sqrt{13} \text{ and } \theta = \tan^{-1}(3/2). \end{aligned}$$

Example 5

Solve $9x_{n+2} + 9x_{n+1} + 2x_n = 0$ with $x_0 = 1$ and $x_1 = 1$.

Example 5
Solution

The given equation can be written as $(9E^2 + 9E + 2)x_n = 0$.

The auxiliary equation is $9E^2 + 9E + 2 = 0$.

It has roots $-1/3$ and $-2/3$.

The general solution is

$$\begin{aligned} x_n &= c_1(-1/3)^n + c_2(-2/3)^n \\ x_0 = 1 &\Rightarrow c_1 + c_2 = 1 \\ x_1 = 1 &\Rightarrow -c_1 - 2c_2 = 3 \end{aligned}$$

Solving we obtain, $c_1 = 5$ and $c_2 = -4$.

Hence the particular solution is $x_n = 5\left(-\frac{1}{3}\right)^n - 4\left(-\frac{2}{3}\right)^n$.

Exercise

(i) $u_{n+2} - 2u_{n+1} - 3u_n = 0$.

(ii) $9y_{n+2} - 6y_{n+1} + y_n = 0$ with $y_0 = 1$ and $y_1 = 1$.

Answer

(i) $u_n = c_1 3^n + c_2(-1)^n$.

(ii) $y_n = 3^n.(1/3)^n$.

- The general solution of a non-homogeneous linear difference equation is found by adding a particular solution called **Particular Integral** (**P.I.**) of the non-homogeneous equation to the general solution called **Complementary Function** (**C.F.**) of the corresponding homogeneous equation.
- Thus, **general solution** = **C.F.** + **P.I.**

Consider equation

$$
\begin{aligned}
f(E)x_n &= g(n) \text{ where} \\
f(E) &= E^r + k_1 E^{r-1} + ... + k_r.
\end{aligned}
$$

Then the particular integral is given by

$$\text{P.I} = \frac{1}{f(E)} . g(n).$$

It can be evaluated by the method of operators. Various cases are given below:

Case (a) ➪ When $g(n) = a^n$, a is a constant

If $f(a) \neq 0 \Rightarrow$ P.I$=\dfrac{1}{f(E)}.a^n = \dfrac{1}{f(a)}.a^n.$

If $f(a) = 0$ then for the equation

(i) $(E - a)x_n = a^n \Rightarrow$ P.I$=\dfrac{1}{E-a}.a^n = n.a^{n-1}.$

(ii) $(E - a)^2 x_n = a^n \Rightarrow$ P.I$=\dfrac{n(n-1)}{2!}.a^{n-2}.$

(iii) $(E - a)^3 x_n = a^n \Rightarrow$ P.I$=\dfrac{n(n-1)(n-2)}{3!}.a^{n-3}$ and so on.

Case (b-I) ➪ When $g(n) = \sin \alpha n$

$$\begin{aligned}
\text{P.I} &= \frac{1}{f(E)}.\sin \alpha n = \frac{1}{f(E)}\left[\frac{e^{i\alpha n} - e^{-i\alpha n}}{2i}\right] \\
&= \frac{1}{2i}\left[\frac{1}{f(E)}.e^{i\alpha n} - \frac{1}{f(E)}.e^{-i\alpha n}\right] \\
&= \frac{1}{2i}\left[\frac{1}{f(E)}.a^{n} - \frac{1}{f(E)}.b^{n}\right] \text{ where } a = e^{i\alpha} \text{ and } b = e^{-i\alpha}.
\end{aligned}$$

Now it is similar to **Case (a)**.

Case (b-II) ➪ When $g(n) = \cos \alpha n$

$$
\begin{aligned}
\text{P.I} &= \frac{1}{f(E)}.\cos \alpha n = \frac{1}{f(E)}\left[\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{f(E)}.e^{i\alpha n} + \frac{1}{f(E)}.e^{-i\alpha n}\right] \\
&= \frac{1}{2}\left[\frac{1}{f(E)}.a^{n} + \frac{1}{f(E)}.b^{n}\right] \text{ where } a = e^{i\alpha} \text{ and } b = e^{-i\alpha}.
\end{aligned}
$$

Now it is similar to **Case (a)**.

Case (c) ➪ When $g(n) = n^p$

$$\text{P.I} = \frac{1}{f(E)}.n^p = \frac{1}{f(1+\Delta)}.n^p$$

The above P.I. is evaluated in two steps.

- Using Binomial theorem, expand $[f(1+\Delta)]^{-1}$ upto the term Δ^p.
- Express n^p in the factorial form and operate the expansion terms on it.

Case (d) ➪ When $g(n) = an.G(n)$, $G(n)$ being a polynomial of degree n and a is a constant.

$$\text{P.I} = \frac{1}{f(E)}.a^n G(n) = a^n.\frac{1}{f(aE)}.G(n)$$

which is evaluated using **Case (c)**.

Example 1

Solve $(E^2 - 5E + 6)x_n = 4^n$.

Auxiliary equation is $E^2 - 5E + 6 = 0$.

Its roots are 2 and 3.

$$\begin{aligned}
\text{C.F.} &= c_1 2^n + c_2 3^n \\
\text{P.I.} &= \frac{1}{E^2 - 5E + 6}.4^n \text{ (put } E = 4) \\
&= \frac{1}{4^2 - 5 \times 4 + 6}.4^n \\
&= \frac{1}{2}.4^n
\end{aligned}$$

Hence the general solution is given by $x_n = c_1 2^n + c_2 3^n + \frac{1}{2}.4^n$

Example 2

Solve $(E^2 - 4E + 4)x_n = 2^n$.

Example 2
Solution

Auxiliary equation is $E^2 - 4E + 4 = 0$.

Its roots are 2 and 2.

$$
\begin{aligned}
\text{C.F.} &= (c_1 + c_2 n).2^n \\
\text{P.I.} &= \frac{1}{E^2 - 4E + 4}.2^n = \frac{1}{(E-2)^2}.2^n \text{ (case fails)} \\
&= \frac{n(n-1)}{2!}.2^{n-2} \text{ ((using (ii))} \\
&= \frac{n(n-1)}{8}.2^n.
\end{aligned}
$$

Hence the general solution is given by

$$x_n = (c_1 + c_2 n).2^n + \frac{n(n-1)}{8}.2^n$$

Example 3

Solve $2x_{n+2} + 3x_{n+1} + x_n = \cos 2n$.

The given equation can be written as

$$(2E^2 + 3E + 1)x_n = \cos 2n.$$

The auxiliary equation is $2E^2 + 3E + 1 = 0$.

It has roots are 1 and 1/2.

C.F.=$c_1(-1)^n + c_2(-1/2)^n$.

Example 3

Solution⇒Cont...

$$
\begin{aligned}
\text{P.I} &= \frac{1}{2E^2+3E+1}.\cos 2n \\
&= \frac{1}{2E^2+3E+1}.\left[\frac{e^{i2n}+e^{-i2n}}{2}\right] \\
&= \frac{1}{2}\left[\frac{1}{2E^2+3E+1}.e^{i2n}+\frac{1}{2E^2+3E+1}.e^{-i2n}\right] \\
&= \frac{1}{2}\left[\frac{1}{2e^{i4}+3e^{i2}+1}.e^{i2n}+\frac{1}{2e^{-i4}+3e^{-i2}+1}.e^{-i2n}\right] \\
&= \frac{1}{2}.\frac{(2e^{-i4}+3e^{-i2}+1)e^{i2n}+(2e^{i4}+3e^{i2}+1)e^{-i2n}}{(2e^{i4}+3e^{i2}+1).(2e^{-i4}+3e^{-i2}+1)} \\
&= \frac{1}{2}.\frac{2\cos(4-2n)+3\cos(2-2n)+\cos 2n}{2\cos 4+9\cos 2+12}
\end{aligned}
$$

Hence the general solution is given by

$$c_1(-1)^n + c_2(-1/2)^n + \frac{1}{2}.\frac{2\cos(4-2n) + 3\cos(2-2n) + \cos 2n}{2\cos 4 + 9\cos 2 + 12}.$$

Example 4

Solve $x_{n+2} - x_{n+1} - 2x_n = n^2$.

The given equation can be written as $(E^2 - E - 2)x_n = n^2$.

The auxiliary equation is $E^2 - E - 2 = 0$.

It has roots are -1 and 2.

Therefore C.F.$=c_1(-1)^n + c_2(2)^n$.

$$
\begin{aligned}
\text{P.I} &= \frac{1}{E^2 - E - 2}.n^2 \\
&= \frac{1}{(1+\Delta)^2 - (1+\Delta) - 2}.n^2 \\
&= \frac{1}{\Delta^2 + \Delta - 2}.n^2 \\
&= -\frac{1}{2}\left[1 - \left(\frac{\Delta^2+\Delta}{2}\right)\right]^{-1}.n^2 \\
&= -\frac{1}{2}\left[1 + \left(\frac{\Delta^2+\Delta}{2}\right) + \left(\frac{\Delta^2+\Delta}{2}\right)^2 + ...\right].n^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[1+\frac{\Delta^2}{2}+\frac{\Delta}{2}+\frac{\Delta^2}{4}+...\right].n^2\\
&= -\frac{1}{2}\left[1+\frac{\Delta}{2}+\frac{3}{4}\Delta^2+...\right].n^2\\
&= -\frac{1}{2}\left[1+\frac{\Delta}{2}+\frac{3}{4}\Delta^2+...\right].(n^{(2)}+n^{(1)})\\
&= -\frac{1}{2}\left[\{n^{(2)}+n^{(1)}\}+\frac{1}{2}\{2n^{(1)}+1\}+\frac{3}{4}\{2.1.n^{(0)}\}\right]\\
&= -\frac{1}{2}(n^2+n+2)
\end{aligned}
$$

NB: Refer following web link for more details on factorial form.
`http://www.fq.math.ca/Scanned/16-1/brousseau.pdf`

Hence the general solution is given by
$c_1(-1)^n + c_2(2)^n - \frac{1}{2}(n^2 + n + 2)$.

Thank you !