Mathematical Modelling-II

(AMT221β/IMT221β)

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About course unit

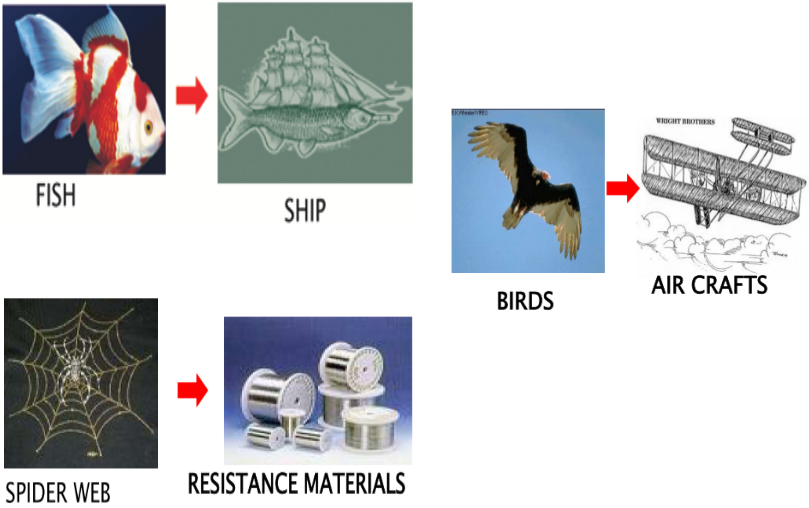
- Course unit: Mathematical Modelling-II(AMT221β/IMT221β)
- Credit value: 2.5
- Number of lecture hours: 30
- Number of tutorial hours: 15
- Method of assessment: End of semester examination
- Attendance: Both tutorial and lecture will be considered

References

1. Numerical Methods, S. Balachandra Rao and C.K. Shantha (519.4RAO)
2. Numerical Methods for Engineers and Scientists, J.N. Sharma (519.4SHA)
3. Numerical Methods, J. Douglas Faires and Richard Burden (519.4FAI)
4. Advance Engineering Mathematics, H.K. Dass (510DAS)
5. http://www.math.ruh.ac.lk/~pubudu/

Introduction to Numerical Solutions of Differential Equations

Nature's influence on scientific investigations



Why do we need mathematical modelling?

1. The mass of the Earth is $5.98 \times 10^{24} kg$. How do we find it? Can we use a balance?

2. The temperature at the very center of the Sun is about 15,000,000°C. How do we measure it? Can we use a thermometer?

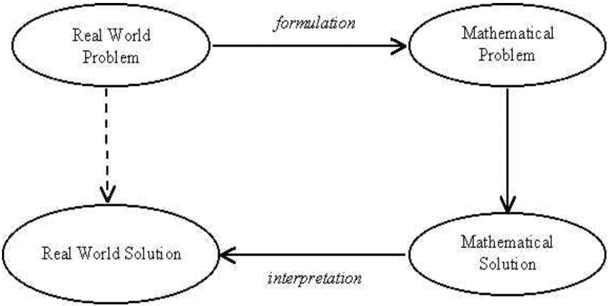
3. A typical adult has a blood volume of approximately between 4.7 and 5 litres. How do we measure it? Can we find a person for this?

Answers to these questions require mathematical modeling !

What is mathematical modelling?

- Mathematical modelling is a process of representing real world problems in mathematical terms and concepts.
- A mathematical model can be considered as a simplification of real world problem into a mathematical form, thereby converting the real world problem into a mathematical problem.
- The mathematical problem can then be solved using whatever known techniques to obtain a mathematical solution.
- This solution is then interpreted and translated into real terms.

What is mathematical modelling?
Cont...



Why do we use differential equations in modelling?

- Many models in real world involve the rate of change of a quantity.
- There is thus a need to incorporate derivatives into the mathematical model.
- These mathematical models are examples of differential equations.

Differential equations

- A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders.
- **Eg:**
  (i) $\dfrac{dx}{dt} = \dfrac{t^2x - x}{x+1}$
  (ii) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 5y = 9x^2$
  (iii) $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$

Types of differential equations

Differential equations can be classified as

1. **Ordinary Differential Equations (ODEs)** and
2. **Partial Differential Equations (PDEs)**.

Ordinary differential equations (ODEs)

- An ordinary differential equation (ODE) is a differential equation in which the unknown function (also known as the dependent variable) is a function of a single independent variable.
- **Eg:**
  (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2$

  (ii) $y' = x^2 - 1$

  (iii) $\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = x + 1$

Partial differential equation (PDEs)

- A partial differential equation (PDE) is a differential equation in which the unknown function is a function of multiple independent variables and the equation involves its partial derivatives.

- **Eg:**

  (i) $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 1$$

  (ii) $$\frac{\partial R}{\partial u} = 3w^2 \frac{\partial R}{\partial w} - 5v \frac{\partial R}{\partial v}$$

Order of a differential equation

- Differential equations, both ordinary and partial, are also classified according to the highest ordered derivative of the unknown function.
- The order of any differential equation is the highest order of the derivative present in the equation.

Order of a differential equation
Example

Classify following differential equations as ordinary or partial and then write down the order.

(i) $\dfrac{\mathrm{d}^3 u}{\mathrm{d}x^2} + \dfrac{\mathrm{d}^2 u}{\mathrm{d}x} = x + 1.$

(ii) $y''' + 3xy'' + \dfrac{6}{x}y = 1 - x^3.$

(iii) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3yx = \dfrac{\sin x}{x}.$

(iv) $\dfrac{\partial R}{\partial u} = 3w^2\dfrac{\partial R}{\partial w} - 5v\dfrac{\partial R}{\partial v}.$

(i) $\dfrac{\mathrm{d}^3u}{\mathrm{d}x^2} + \dfrac{\mathrm{d}^2u}{\mathrm{d}x} = x + 1 \Rightarrow$ ODE of oder 3.

(ii) $y''' + 3xy'' + \dfrac{6}{x}y = 1 - x^3 \Rightarrow$ ODE of oder 3.

(iii) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3yx = \dfrac{\sin x}{x} \Rightarrow$ ODE of oder 1.

(iv) $\dfrac{\partial R}{\partial u} = 3w^2\dfrac{\partial R}{\partial w} - 5v\dfrac{\partial R}{\partial v} \Rightarrow$ PDE of oder 1.

Linear and non-linear

- Both ordinary and partial differential equations are broadly classified as linear and nonlinear.
- If the equation is linear, then the variables and their derivatives must always appear in first power (products are not allowed).
- An ordinary differential equation is **linear** if it is written in the following general form:

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \ldots + a_1(x)y + a_0(x) = f(x).$$

State whether following differential equations are linear or non linear.

(i) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$

(ii) $y'' + 5y' + 3y = 0$

(iii) $y' + y^2 = 0$

(iv) $x' + 1/x = 0$

(v) $x'' + \cos x = 0$

(vi) $xx' = 1$

Linear and non-linear

Example⇒Solution

(i) $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0 \Leftarrow$ linear

(ii) $y'' + 5y' + 3y = 0 \Leftarrow$ linear

(iii) $y' + y^2 = 0 \Leftarrow$ non-linear because y^2 is not a first power

(iv) $x' + 1/x = 0 \Leftarrow$ non-linear because $1/x$ is not a first power

(v) $x'' + \cos x = 0 \Leftarrow$ non-linear because $\cos x$ is not a first power

(vi) $xx' = 1 \Leftarrow$ non-linear because x' is not multiplied by a constant

Solutions of differential equations

- In the case of an ordinary differential equation, a solution is a function defined on some interval J with the property that the equation reduces to an identity when the function is substituted into the equation.

- In the case of a partial differential equation, a solution is a function defined on some domain D in two or higher dimensional space with the property that the equation reduces to an identity when the function is substituted into the equation.

Show that $y = x^2 + 2x^3$ is a solution of the following second-order ordinary differential equation:

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^3.$$

- Suppose you have a mathematical model and you want to understand its behavior.
- That is, you want to find a solution to the set of equations.
- The best is when you can use calculus, trigonometry, and other mathematical techniques to write down the solution.
- Now you know absolutely how the model will behave under any circumstances.
- This is called the **analytic solution**, because you used analysis to figure it out.

- Taking analytical solutions works only for simple models.
- For more complex models, the math becomes much too complicated.
- Then you turn to numerical methods of solving the equations, such as the Runge-Kutta method.

- For a differential equation that describes behavior over time, the numerical method starts with the initial values of the variables, and then uses the equations to figure out the changes in these variables over a very brief time period.
- Its only an approximation, but it can be a very good approximation under certain circumstances.
- A computer must be used to perform the thousands of repetitive calculations involved.
- The result is a long list of numbers, not an equation.

Thank you !