Mathematics for Biology

MAT1142

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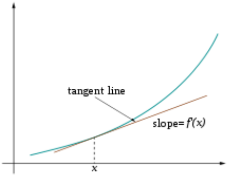
Pre-Requisities for Maxima and Minima

Stationary points

- The derivative $\mathrm{d}y/\mathrm{d}x$ of a function $y = f(x)$ tell us a lot about the shape of a curve.
- The derivative $\mathrm{d}y/\mathrm{d}x$ is the **slope of the tangent** to the curve $y = f(x)$ at the point x.
- The **stationary points** of a function are those points where the derivative of the function is zero.



Stationary points
Examples

Find the stationary points of the following functions.

(i) $f(x) = 3x^2 + 2x - 9$

(ii) $f(x) = x^3 - 6x^2 + 9x - 2$

(iii) $f(t) = 16 - 6t - t^2$

(iv) $y = 3u^2 - 4u + 7$

Stationary points

Exercise

Find the stationary points of the following functions.

(i) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$

(ii) $y = x^3 - 6x^2 - 15x + 16$

(iii) $f(s) = \frac{1}{3}s^2 - 12s + 32$

(iv) $y = x^3 - 12x + 12$

Motivative example for second derivative

- The height $H(t)$ in metres of a ball thrown vertically at $20ms^{-1}$, was given by,

$$H(t) = 20t - 10t^2.$$

- The **velocity** of the ball, $v\ ms^{-1}$, after t seconds, was given by,

$$v(t) = \frac{\mathrm{d}H}{\mathrm{d}t} = 20 - 20t.$$

Motivative example for second derivative

Cont...

- The rate of change of velocity with time, which is the **acceleration**, is then given by $a(t)$, where,

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -20ms^{-2}.$$

What is second derivative of a function?

- The acceleration was derived from $H(t)$ by two successive differentiations.
- The resulting function, which in this case is $a(t)$, is called the second derivative of $H(t)$ with respect to t.
- It can be written mathematically as,

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 H}{\mathrm{d}t^2}.$$

Higher derivatives of functions

Examples

Find the first and the second derivatives of the following functions.

(i) $f(x) = x^3$

(ii) $y = x^2 + 5$

(iii) $f(x) = 2x^3 + 3x^2 - 7x + 5$

(iv) $f(x) = x^3 - 6x^2 + 9x - 2$

(v) $y = x^3 + e^x$

(vi) $y = \sin x + \cos x$

Higher derivatives of functions
Exercise

Find the first and the second derivatives of the following functions.

(i) $f(t) = 16 - 6t - t^2$

(ii) $y = 3v^2 - 4v + 7$

(iii) $y = \frac{1}{5}x^5 - x^4 - 3x^3 + 2x + 4$

(iv) $y = 4x^3 - 12x^2 - 5x + 16$

(v) $y = x^5 + e^x + \sin x + 12$

(vi) $y = \tan x + 5\cos x + 9$

Maxima and Minima

Introduction

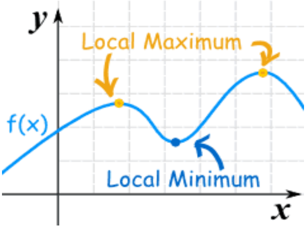
- When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of **variables**.
- Then, **functions** are used to describe the ways in which these variables change.
- A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values, its turning points.

How do we locate maximum and minimum points?

- Drawing a graph of a function using a computer graph plotting package will reveal behaviour of the function.
- But if we want to know the precise location of maximum and minimum points, we need to turn to algebra and differential calculus.
- In this section we look at how we can find maximum and minimum points in this way.

Local maximum and local minimum

- The **local maximum** and **local minimum** (plural: maxima and minima) of a function, are the largest and smallest value that the function takes at a point within a given interval.
- It may not be the minimum or maximum for the whole function, but **locally** it is.

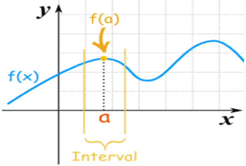


Local maximum and local minimum

Local maximum

- To define a local maximum, we need to consider an interval.
- Then a **local maximum** is the point where, the height of the function at **a** is greater than (or equal to) the height anywhere else in that interval.
- Or, more briefly:

$$\mathbf{f(a) \geq f(x) \text{ for all } x \text{ in the interval}}.$$



Local maximum and local minimum

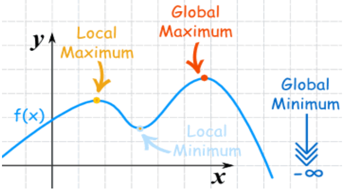
Local minimum

- To define a local minimum, we need to consider an interval.
- Then a **local minimum** is the point where, the height of the function at **a** is lowest than (or equal to) the height anywhere else in that interval.
- Or more briefly:

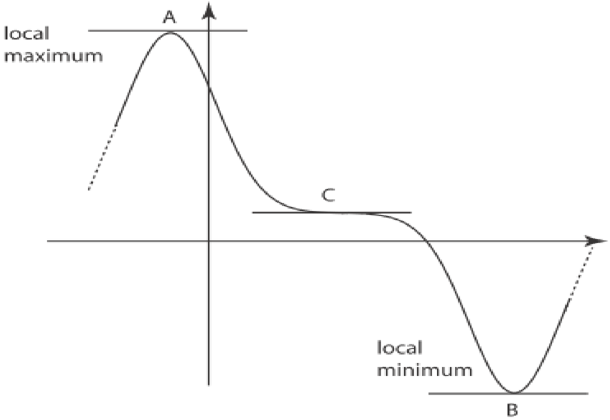
$$\mathbf{f(a) \leq f(x)} \textbf{ for all x in the interval}.$$

Global (or Absolute) Maximum and Minimum

- The maximum or minimum over the entire function is called an **absolute** or **global** maximum or minimum.
- There is **only one** global maximum.
- And also there is **only one** global minimum.
- But there can be **more than one** local maximum or minimum.



Locating maximum and minimum of any function $y(x)$



Locating maximum and minimum of any function $y(x)$

Tangents to the graph

- If we draw tangents to the graph at the points A, B and C, note that these are parallel to the x axis.
- They are horizontal.
- This means that at each of the points A, B and C the slope of the graph is zero.

Locating maximum and minimum of any function $y(x)$

Stationary points

- We know that the slope of a graph is given by $\mathrm{d}y/\mathrm{d}x$.
- Consequently, $\mathrm{d}y/\mathrm{d}x = 0$ at points A, B and C.
- Therefore all of these points are stationary points.

Locating maximum and minimum of any function $y(x)$

Turning point

- Notice that at points A and B the curve actually turns.
- These two stationary points are referred to as **turning points**.
- Point C is not a turning point and it is an **inflection point**.
- Because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

Locating maximum and minimum of any function $y(x)$

Remark

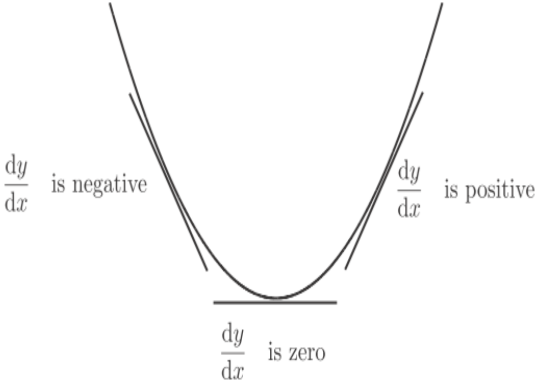
- So all turning points are stationary points.
- But not all stationary points are turning points.
- In other words, there are points for which $\mathrm{d}y/\mathrm{d}x = 0$ which are not turning points.

Locating maximum and minimum of any function $y(x)$

Identification of local maximum, local minimum and inflection points

- **Point A** is called a **local maximum** because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function.
- **Point B** is called a **local minimum** because in its immediate area it is the lowest point, and so represents the least, or minimum value of the function.
- **Point C** is called a **inflection point**, because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

The graph of a minimum point



The graph of a minimum point

Behavior of $\mathrm{d}y/\mathrm{d}x$ around a minimum point

- Notice that to the left of the minimum point, $\mathrm{d}y/\mathrm{d}x$ is negative because the tangent has negative slope.
- At the minimum point, $\mathrm{d}y/\mathrm{d}x = 0$.
- To the right of the minimum point $\mathrm{d}y/\mathrm{d}x$ is positive, because here the tangent has a positive slope.
- So, $\mathrm{d}y/\mathrm{d}x$ goes from negative, to zero, to positive as x increases.
- In other words, $\mathrm{d}y/\mathrm{d}x$ must be increasing as x increases.

Distinguishing minimum points from stationary points

- In fact, we can use this observation, once we have found a stationary point, to check if the point is a minimum.
- If $\mathrm{d}y/\mathrm{d}x$ is increasing near the stationary point then that point must be minimum.

Distinguishing minimum points from stationary points

Behavior of second derivative around a minimum point

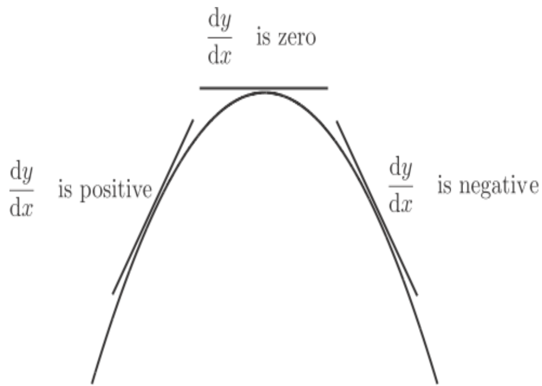
- The derivative of $\mathrm{d}y/\mathrm{d}x$ called the **second derivative**, is written $\mathrm{d}^2y/\mathrm{d}x^2$.
- If $\mathrm{d}^2y/\mathrm{d}x^2$ is positive then we will know that $\mathrm{d}y/\mathrm{d}x$ is increasing.
- So we will know that the stationary point is a minimum.

Distinguishing minimum points from stationary points

Remark

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ there, then that point must be a minimum.

The graph of a maximum point



The graph of a maximum point

Behavior of $\mathrm{d}y/\mathrm{d}x$ around a maximum point

- Notice that to the left of the maximum point, $\mathrm{d}y/\mathrm{d}x$ is positive because the tangent has positive slope.
- At the maximum point, $\mathrm{d}y/\mathrm{d}x = 0$.
- To the right of the maximum point $\mathrm{d}y/\mathrm{d}x$ is negative, because here the tangent has a negative slope.
- So, $\mathrm{d}y/\mathrm{d}x$ goes from positive, to zero, to negative as x increasing.
- In other words, $\mathrm{d}y/\mathrm{d}x$ must be decreasing as x increases.

Distinguishing maximum points from stationary points

- In fact, we can use this observation, once we have found a stationary point, to check if the point is a maximum.
- If $\mathrm{d}y/\mathrm{d}x$ is decreasing near the stationary point then that point must be maximum.

Distinguishing maximum points from stationary points

Behavior of second derivative around a maximum point

- The derivative of $\mathrm{d}y/\mathrm{d}x$ called the **second derivative**, is written $\mathrm{d}^2y/\mathrm{d}x^2$.
- If $\mathrm{d}^2y/\mathrm{d}x^2$ is negative then we will know that $\mathrm{d}y/\mathrm{d}x$ is decreasing.
- So we will know that the stationary point is a maximum.

Distinguishing maximum points from stationary points

Remark

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ there, then that point must be a maximum.

Summary of distinguishing stationary points

- We can locate the positions of stationary points by looking for points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- As we have seen, it is possible that some such points will not be turning points.
- We can calculate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at each stationary point.

Summary of distinguishing stationary points

Cont...

- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is positive then the stationary point is a **minimum turning point**.
- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is negative, then the point is a **maximum turning poin**.
- If $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ this second derivative test does not give us useful information.

Example 1

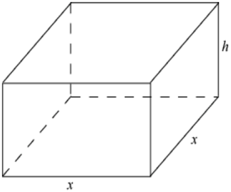
Find the stationary points of the functions and hence determine the nature of these points.

(a) $y = x^2 - 2x + 3$

(b) $y = x^3 - 3x^2 - 9x + 3$

Example 2

The material for the square base of a rectangular box with open top costs 27 cents per square *cm* and for the other faces costs $13\frac{1}{2}$ cents per square *cm*. Find the dimensions of such a box of maximum volume which can be made for Rs 65.61.



Example 3

We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard which is already fenced, so we only need to build a new fence along the other 3 sides of the rectangle (see figure). If we have 80 feet of fencing available, what dimensions should the garden have in order to enclose the largest possible area?

Exercise

Determine the nature of the stationary points of the following functions.

(a) $y = 16 - 6u - u^2$

(b) $y = 3x^2 - 4x + 7$

Thank You