Mathematics for Biology

MAT1142

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Logarithms

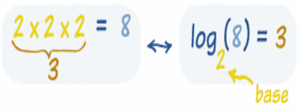
Why do we need logarithms?

- Sometimes you only care about how big a number is relative to other numbers.
- The **Richter**, **decibel**, and **pH** scales are good examples for relative representations.
- An earthquake that measures 5.0 on the **Richter** scale has a shaking amplitude 10 times larger than one that measures 4.0.
- To do such relative representations we need logarithms.
- Logarithms answer the question **"To what power to I need to raise X to get Y?"**

Motivative example

How many 2's do we multiply to get 8?

- The number of 2s we need to multiply to get 8 is 3.
- That is $2 \times 2 \times 2 = 8$.
- It can be written down as $\log_2(8) = 3$.
- Therefore the logarithm is 3.
- $\log_2(8) = 3$ is called as the logarithm of 8 with base 2 is 3.



Examples

(i) What is $\log_{10}(100)$?

(ii) What is $\log_5(125)$?

(iii) What is $\log_5(625)$?

(iv) What is $\log_2(128)$?

(v) What is $\log_3(81)$?

(vi) What is $\log_2(1/8)$?

Definition

- The logarithm of a number $\mathbf{x}$ to a base $\mathbf{b}$ is just the exponent you put onto $\mathbf{b}$ to make the result equal $\mathbf{x}$.
- Since $4^2 = 16$, we know that 2 (the power) is the logarithm of 16 to base 4. Symbolically, $\log_4(16) = 2$.
- More generically, if $\mathbf{x} = \mathbf{b}^{\mathbf{y}}$, then we say that $\mathbf{y}$ is "the logarithm of $\mathbf{x}$ to the base $\mathbf{b}$". In symbols, $\mathbf{y} = \log_{\mathbf{b}}(\mathbf{x})$.

$$\mathbf{x} = \mathbf{b}^{\mathbf{y}} \iff \mathbf{y} = \log_{\mathbf{b}}(\mathbf{x})$$

Remark 1

- The base of a logarithm should be a positive number.
- We define only the logarithm of positive numbers.

Remark 2

- We know that anything to the zero power is 1.
- That is $b^0 = 1$.
- By definition of logs we have,

$$\log_{\mathbf{b}} \mathbf{1} = \mathbf{0} \textbf{ for any base b}.$$

Remark 3

- We know that the first power of any number is just that number.
- That is $b^1 = b$.
- Again, turn that around to logarithmic form we have,

$$\log_{\mathbf{b}} \mathbf{b} = \mathbf{1} \textbf{ for any base } \mathbf{b}.$$

Properties of logarithms

1. $\log_a(mn) = \log_a(m) + \log_a(n)$
2. $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
3. $\log_a m^n = n \log_a m$

Examples

Simplify following expressions.

(i) $\log_a 3 + \log_a 4.$

(ii) $\log_a 6 - \log_a 2.$

(iii) $\log_a 2 + \log_a 6 - \log_a 4.$

(iv) $2\log_a 3 + \log_a 2.$

(v) $\frac{1}{2}\log_a 4 - \log_a 6.$

(vi) $\frac{\log_a 125}{\log_a 5}.$

Common logarithms

- Any positive number is suitable as the base of logarithms, but base 10 is used more than any others.
- The logarithm with base 10 is called as **common logarithm**.
- Sometimes you will see a logarithm written without a base, like this: $\log 1000$.
- This usually means that the base is really 10.
- **Eg:**

$$\log 1000 = \log_{10} 1000 = 3$$

Common logarithms

Examples

(i) $\log_{10} 100$

(ii) $\log 1000$

(iii) $\log 0.1$

(iv) $\log 0.001$

(v) $\log\left(\frac{1}{\sqrt{10}}\right)$

Natural logarithms

- The logarithm with base e is called as **natural logarithm**.
- Numerically, e is about 2.7182818284.
- Its an irrational number.

$$\log_e x \Longleftrightarrow \ln x$$

- **Eg:**

$$\ln(7.389) = \log_e(7.389) \simeq \log_e(2.71828^2) = 2$$

Natural logarithms

Examples

(i) $\ln e^2$

(ii) $\ln \sqrt{e}$

(iii) $e^{2\ln 4}$

(iv) $\frac{1}{2}(4\ln 2 - 2\ln 5)$

Changing the base

To change the log from base **b** to another base (call it **a**), we can use the following formula.

$$\log_{\mathbf{a}} \mathbf{m} = \frac{\log_{\mathbf{b}} \mathbf{m}}{\log_{\mathbf{b}} \mathbf{a}}$$

Examples

(i) Evaluate $\log_2 10$

(ii) Evaluate $\log_7 2$

(iii) Evaluate $\log_3 9$

(iv) $5^x = 4$, find the value of x.

(v) $4^x - 6(2^x) - 16 = 0$, find the value of x.

Remark

$$
\begin{aligned}
\ln x &= \log_e x \\
\ln x &= \frac{\log_{10} x}{\log_{10} e} \\
\ln x &= \frac{\log_{10} x}{0.4343} \\
\ln \mathbf{x} &\mathbf{=} \mathbf{2.302555} \log_{\mathbf{10}} \mathbf{x}
\end{aligned}
$$

Example

Let $H = 30(1 - e^{-0.3t})$. It is known that when $t = 0$ the value of $H = 0$. You are given that $H = 15cm$ after certain time T. Find the value of T.

Thank You