Mathematics for Biology

MAT1142

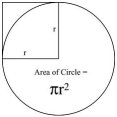
A.W.L. Pubudu Thilan

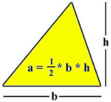
Department of Mathematics
University of Ruhuna

Introduction to Integration

Why do we need integration?

- If we know radius r of a circle then we can calculate its area.
- The area of the circle with radius r is given by πr^2.
- If we know lenght of the base b and the height h of a triangle then we can calculate its area.
- The area of the triangle is given by $\frac{1}{2}bh$.



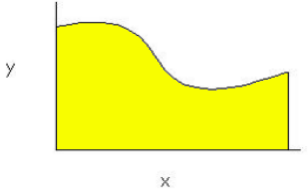


Why do we need integration?
Cont...

- In similar manner we can calculate the area of a square, rectangle, and other regular polygons.
- Only thing we need to do is "subsitution of known measurements into corresponding formulas".

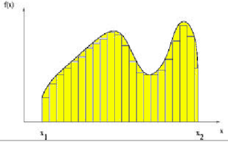
Why do we need integration?
Cont...

- A serious problem arises when one wishes to calculate the area of an irregular curve.
- Such shapes cannot easily be plugged into a convenient formal and the area produced.
- The integration plays an important role in calculating area of such irregular shapes.



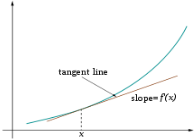
History of integration

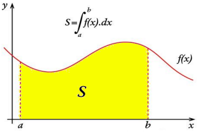
- The first steps towards integral calculus actually began in ancient Greece.
- In the third century B.C., Aristotle became interested in areas defined by certain curves.
- He used rectangles to approximate these regions.
- Then used smaller and smaller rectangles, so that the approximation became better and better.



Integration as differentiation in reverse

- The integration can be considered as **anti differentiation**.
- That means integration is the reverse side of differentiation.
- When you differentiate an equation you get the **slope**.
- When you integrate you get the **area** between equation and the *x*-axis.



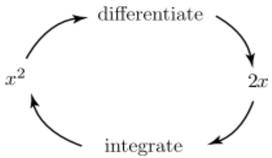


Integration as differentiation in reverse
Cont...

- The integral or anti-derivative of a function is another function such that the derivative of that function is equal to the original function.
- That is if $G(x)$ is the anti-derivative of $F(x)$, then the derivative of $G(x)$ is equal to $F(x)$.

Integration as differentiation in reverse
Example

- Suppose we differentiate the function $y = x^2$.
- Then we obtain the derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$.
- Integration reverses this process and we say that the integral of $2x$ is x^2.



- The situation is just a little more complicated because there are lots of functions we can differentiate to give $2x$.
- Example for such functions are:
  $x^2+5,\ x^2-13,\ x^2+\dfrac{1}{5},\ x^2+100.$
- All these functions have the same derivative, $2x$.

- When we differentiate the constant term we obtain zero.
- Consequently, when we reverse the process, we have no idea what the original constant term might have been.
- So we include in our answer an unknown constant, c.
- That constant c is called as **constant of integration**.
- We state that the integral of $2x$ is $x^2 + c$.

Notations used in integration

- When we want to integrate function $f(x)$ we use a special notation: $\displaystyle\int f(x)\,\mathrm{d}x$.
- The symbol $\displaystyle\int$ is known as an **integral sign**.

Terms used in integration

- Along with the integral sign there is a term of the form $\mathrm{d}x$, which must always be written, and which indicates the **variable** involved, in this case x.
- We say that $2x$ is being **integrated** with respect to x.
- The function being integrated is called the **integrand**.

$$\int 2x \ \mathrm{d}x = x^2 + c$$

integral sign

this term is called the integrand

there must always be a term of the form $\mathrm{d}x$

constant of integration

Indefinite integrals

Introduction

- A integral of the form $\int f(x)\,\mathrm{d}x$ is called an **indefinite integral**.
- The indefinite integral of $f(x)$ is a **function**.
- That answers the question, "What function when differentiated gives $f(x)$?"

A table of integrals for some basic functions

Function $f(x)$	**Indefinite integral** $\int f(x)\,\mathrm{d}x$
constant, k	$kx + c$
x^n	$\frac{1}{n}x^n + c, \; n \neq -1$
$\frac{1}{x}$	$\ln\|x\| + c$

A table of integrals for some basic functions
Examples

Integrate each of the following functions:

(i) $\int 12 \, \mathrm{d}x$

(ii) $\int x^6 \, \mathrm{d}x$

(iii) $\int x^{1/2} \, \mathrm{d}x$

(iv) $\int x^{-5} \, \mathrm{d}x$

(v) $\int \frac{1}{x^3} \, \mathrm{d}x$

(vi) $\int t^3 \, \mathrm{d}t$

(vii) $\int 4 \, \mathrm{d}t$

(viii) $\int \sqrt{u} \, \mathrm{d}u$

(xi) $\int x^{100} \, \mathrm{d}x$

(x) $\int \frac{1}{\sqrt{v}} \, \mathrm{d}v$

A table of integrals for trigonometric functions

Function $f(x)$	Indefinite integral $\int f(x)\,\mathrm{d}x$
$\sin x$	$-\cos x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cos x$	$\sin x + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert + c$

A table of integrals for trigonometric functions
Examples

Integrate each of the following functions:

(i) $\displaystyle\int \cos 5x \, \mathrm{d}x$

(ii) $\displaystyle\int \cos 4t \, \mathrm{d}t$

(iii) $\displaystyle\int \sin 3x \, \mathrm{d}x$

(iv) $\displaystyle\int \cos 3w \, \mathrm{d}w$

A table of integrals for exponential functions

Function $\mathbf{f(x)}$	Indefinite integral $\int \mathbf{f(x)}\mathrm{d}\mathbf{x}$
e^x	$e^x + c$
e^{-x}	$-e^{-x} + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$

A table of integrals for exponential functions
Examples

Integrate each of the following functions:

(i) $\displaystyle\int e^{3x}\,\mathrm{d}x$

(ii) $\displaystyle\int e^{2t}\,\mathrm{d}t$

(iii) $\displaystyle\int e^{x/4}\,\mathrm{d}x$

(iv) $\displaystyle\int \frac{1}{e^{3w}}\,\mathrm{d}w$

Rules of integration

- Above tables consists of integrals of some common functions.
- But we can not integrate all functions directly as above.
- **Eg:** $\displaystyle\int x \sin 3x \, \mathrm{d}x, \int e^{4x} \tan \sqrt{x} \, \mathrm{d}x, \int (e^x + x^3) \, \mathrm{d}x$.
- To deal with such complicated functions, we have to introduce some rules.
- Let us consider some rules used in integration.

Rules of integration
The integral of $kf(x)$ where k is a constant

A constant term in an integral can be taken out of the integral sign as follows:

$$\int \mathbf{k}\, \mathbf{f}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathbf{k} \int \mathbf{f}(\mathbf{x})\, \mathrm{d}\mathbf{x}$$

Find the integrals of following functions:

(i) $\displaystyle\int 4x \, \mathrm{d}x$

(ii) $\displaystyle\int 5x^3 \, \mathrm{d}x$

(iii) $\displaystyle\int 3t \, \mathrm{d}t$

(iv) $\displaystyle\int 3\sin x \, \mathrm{d}x$

(v) $\displaystyle\int 2e^x \, \mathrm{d}x$

(vi) $\displaystyle\int \frac{4}{x^2} \, \mathrm{d}x$

(vii) $\displaystyle\int -\frac{1}{\sqrt{2}}x^8 \, \mathrm{d}x$

(viii) $\displaystyle\int 4e^{2x} \, \mathrm{d}x$

(ix) $\displaystyle\int \frac{1}{\sqrt{3x}} \, \mathrm{d}x$

(x) $\displaystyle\int 2\sec^2 x \, \mathrm{d}x$

Find the integrals of following functions:

(i) $\int 8x \,\mathrm{d}x$

(ii) $\int 2x^4 \,\mathrm{d}x$

(iii) $\int 12t^2 \,\mathrm{d}t$

(iv) $\int 9\cos x \,\mathrm{d}x$

(v) $\int 5e^x \,\mathrm{d}x$

(vi) $\int \frac{3}{x^4} \,\mathrm{d}x$

(vii) $\int -\frac{1}{\sqrt{5}}x^6 \,\mathrm{d}x$

(viii) $\int 5e^{3x} \,\mathrm{d}x$

(ix) $\int \frac{1}{\sqrt{13x}} \,\mathrm{d}x$

(x) $\int \frac{1}{4}\cos 4x \,\mathrm{d}x$

If we need to integrate the sum or difference of two functions, instead of that we can integrate each term separately as follows to get the required result:

$$\int [\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})] \ \mathrm{d}\mathbf{x} = \int \mathbf{f}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \int \mathbf{g}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$
$$\int [\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})] \ \mathrm{d}\mathbf{x} = \int \mathbf{f}(\mathbf{x}) \, \mathrm{d}\mathbf{x} - \int \mathbf{g}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Find the integrals of following functions:

(i) $\int (2x+3)\,\mathrm{d}x$

(ii) $\int (4x^3+2x+5)\,\mathrm{d}x$

(iii) $\int (2t^2+6t+8)\,\mathrm{d}t$

(iv) $\int (5\sin x+4x)\,\mathrm{d}x$

(v) $\int \left(e^x+x^3\right)\,\mathrm{d}x$

(vi) $\int \left(x^3+\frac{2}{x^3}\right)\,\mathrm{d}x$

(vii) $\int \left(9x^3-\frac{4}{x^3}\right)\,\mathrm{d}x$

(viii)
$$\int [\sec^2 x-\sin x+4x^2]\,\mathrm{d}x$$

(ix)
$$\int [2\sin 2x+3(x+1)^2]\,\mathrm{d}x$$

(x) $\int (x+4)^2\,\mathrm{d}x$

Find the integrals of following functions:

(i) $\displaystyle\int (x^6 + 5x + 9)\, \mathrm{d}x$

(ii) $\displaystyle\int \left(2x^2 - \frac{1}{x^2} + x\right)\, \mathrm{d}x$

(iii) $\displaystyle\int (4t^3 - 5t + 6)\, \mathrm{d}t$

(iv) $\displaystyle\int (2x-1)^2\, \mathrm{d}x$

(v) $\displaystyle\int (3x^3 + x^{-3} + 3)\, \mathrm{d}x$

(vi) $\displaystyle\int \left(\frac{4}{x^3} - \frac{1}{x^2} - x^2\right)\, \mathrm{d}x$

(vii) $\displaystyle\int \left(2x^{5/2} - x^{-2/5}\right)\, \mathrm{d}x$

(viii) $\displaystyle\int (5x^4 - 3x^2 + 7)\, \mathrm{d}x$

(ix) $\displaystyle\int (4x^{-3} + x^{-4} + 1)\, \mathrm{d}x$

(x) $\displaystyle\int \left(\frac{1}{2}x - \frac{2}{\sqrt{x}} - 1\right)\, \mathrm{d}x$

- In here before evaluating the given integral we do a substitution to simplify it.
- A more complicated part of the function we are trying to integrate has to be replaced by a new variable (say u).
- The choice of which substitution to make often relies upon experience.

Rules of integration
Integration by substitution⇒Examples

(i) $\displaystyle\int (2x+1)^6 \, \mathrm{d}x$

(ii) $\displaystyle\int x^2 \sin(x^3+1) \, \mathrm{d}x$

(iii) $\displaystyle\int 3t^2 e^{t^3} \, \mathrm{d}t$

(iv) $\displaystyle\int (x^2+6)^{3/2} \, \mathrm{d}x$

(v) $\displaystyle\int \frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x$

Rules of integration
Integration by parts

- In most of the situations we have to deal with functions arise as products of other functions.
- For example, we may be asked to integrate functions of the form below.

$$\int x^2 \sin x \, \mathrm{d}x$$

- In above, the integrand is the product of the functions x^2 and $\sin x$.
- It is difficult to integrate these kind of functions directly.
- We can use **integration by parts** method to integrate these kind of functions.

Let $y = uv$. If we use product formula to differntiate $y = uv$, then we have,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \\
u\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}(uv)}{\mathrm{d}x} - v\frac{\mathrm{d}u}{\mathrm{d}x}
\end{aligned}
$$

Now integrate both sides:

$$\begin{aligned}\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x &= \int \frac{\mathrm{d}(uv)}{\mathrm{d}x}\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \\ \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x &= uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x\end{aligned}$$

This is the formula known as **integration by parts**.

(i) $\displaystyle\int xe^x \, \mathrm{d}x$

(ii) $\displaystyle\int x\sin x \, \mathrm{d}x$

(iii) $\displaystyle\int e^x \sin x \, \mathrm{d}x$

(iv) $\displaystyle\int e^{ax} \sin bx \, \mathrm{d}x$

(v) $\displaystyle\int 2x^2 e^x \, \mathrm{d}x$

(i) $\displaystyle\int x \cos 4x \, \mathrm{d}x$

(ii) $\displaystyle\int e^{x} \cos x \, \mathrm{d}x$

(iii) $\displaystyle\int x^{2} \cos x \, \mathrm{d}x$

(iv) $\displaystyle\int x^{2} e^{3x} \, \mathrm{d}x$

(i) $\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + c$

(ii) $\frac{1}{2}e^{x}(\cos x + \sin x) + c$

(iii) $x^2\sin x + 2x\cos x - 2\sin x + c$

(iv) $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + c$

If $f(x)$ is a function of x and $f'(x)$ is the derivative of $f(x)$, then

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c.$$

(i) $\displaystyle\int \frac{2}{2x+5}\,\mathrm{d}x$

(ii) $\displaystyle\int \frac{4}{4x+9}\,\mathrm{d}x$

(iii) $\displaystyle\int \frac{1}{4x+9}\,\mathrm{d}x$

(iv) $\displaystyle\int \frac{1}{-2x+7}\,\mathrm{d}x$

(v) $\displaystyle\int \frac{2x}{x^2+7}\,\mathrm{d}x$

(vi) $\displaystyle\int \frac{x}{x^2+7}\,\mathrm{d}x$

(vii) $\displaystyle\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x$

(viii) $\displaystyle\int \frac{4x-4}{x^2-2x+1}\,\mathrm{d}x$

(ix) $\displaystyle\int \frac{e^x}{1+e^x}\,\mathrm{d}x$

(x) $\displaystyle\int \frac{e^{-x}}{5+e^{-x}}\,\mathrm{d}x$

(i) $\displaystyle\int \frac{3}{3x+8}\,\mathrm{d}x$

(ii) $\displaystyle\int \frac{2}{2x-6}\,\mathrm{d}x$

(iii) $\displaystyle\int \frac{1}{2x+9}\,\mathrm{d}x$

(iv) $\displaystyle\int \frac{1}{-3x+9}\,\mathrm{d}x$

(v) $\displaystyle\int \frac{2x}{x^2+3}\,\mathrm{d}x$

(vi) $\displaystyle\int \frac{x}{x^2-4}\,\mathrm{d}x$

- This may be a very important step in integrating the more complicated fraction.
- The partial fraction method is used to **breaking apart** fractions with polynomials in them.
- The partial fractions are each simpler.
- So it is easy to integrate these simpler fractions than integrating original more complicated fractions.

- A **rational function** has the form $\dfrac{p(x)}{q(x)}$.
- Where $p(x)$ and $q(x)$ are polynomials.
- A rational function is called **proper** if the degree of the numerator is less than the degree of the denominator.
- If the degree of the numerator is equal or greater than the degree of the denominator, a rational function is called **improper**.

Integration by partial fractions
Examples for proper and improper rational function

Rational function $\frac{p(x)}{q(x)}$	Is proper?
$\frac{x+2}{(x-1)(x-2)}$	Yes
$\frac{x^2-5x+9}{x^2-3x+7}$	No
$\frac{x^3-5x^2+9}{x^2-3x+7}$	No
$\frac{6}{t^3-3t+7}$	Yes
$\frac{x^7-5x^2+9}{x^5-3x^4+7x+9}$	No

Integration by partial fractions
Condition for partial fractions

- Partial fractions can be directly applied for proper rational functions.
- But if the rational function is improper, first we have to divide numerator polynomial by its denominator polynomial.

- If we have improper rational function (i.e. degree of $p(x) >$ degree $q(x)$), then

$$\frac{p(x)}{q(x)} = n(x) + \frac{r(x)}{q(x)}.$$

- Where $n(x)$ being a polynomial and $r(x)$ being a polynomial of degree strictly smaller than the degree of $q(x)$.
- Now $\dfrac{r(x)}{q(x)}$ is a proper rational function and partial fractions can be applied for that.

- By considering a common denominator, fractions with different denominators can be combined into one fraction.
- For example $\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.
- This technique can be applied for denominators with variables as well.

$$
\begin{aligned}
\frac{3}{(x+1)} + \frac{2}{(x+3)} &= \frac{3(x+3)}{(x+1)(x+3)} + \frac{2(x+1)}{(x+1)(x+3)} \\
&= \frac{5x+11}{(x+1)(x+3)}
\end{aligned}
$$

Integration by partial fractions
Concept behind partial fractions⇒Cont...

- Suppose we need to decompose the above rational fraction into separate fractions.
- To do that we would reverse the above steps.
- But how do we determine that we should use 3 and 2 for numerators for the individual fractions?
- The method known as partial fraction helps to answer that.

There are many different ways to decompose a rational function into partial fractions. In here we consider three different ways to find partial fractions.

1. Linear factors in denominator
2. Repeated factor in the denominator
3. Quadratic factor in the denominator

(i) Find the partial fractions of $\dfrac{x+35}{x^2-25}$. Use those partial fractions to evaluate $\displaystyle\int \frac{x+35}{x^2-25}\,\mathrm{d}x$.

(ii) Find the partial fractions of $\dfrac{4x-3}{x^2-5x+6}$. Use those partial fractions to evaluate $\displaystyle\int \frac{4x-3}{x^2-5x+6}\,\mathrm{d}x$.

(i) Find the partial fractions of $\dfrac{3x+5}{x^2+3x+2}$. Use those partial fractions to evaluate $\displaystyle\int \frac{3x+5}{x^2+3x+2}\,\mathrm{d}x$.

$$\left[\text{Answer:}\frac{3x+5}{x^2+3x+2} = \frac{1}{(x+2)} + \frac{2}{(x+1)}\right]$$

(ii) Find the partial fractions of $\dfrac{5x+10}{(x+1)(x+6)}$. Use those partial fractions to evaluate $\displaystyle\int \frac{5x+10}{(x+1)(x+6)}\,\mathrm{d}x$.

$$\left[\text{Answer:}\frac{5x+10}{(x+1)(x+6)} = \frac{1}{(x+1)} + \frac{1}{(x+6)}\right]$$

(i) Find the partial fractions of $\dfrac{3x+1}{(x+1)^2}$. Use those partial fractions to evaluate $\displaystyle\int \frac{3x+1}{(x+1)^2}\,\mathrm{d}x$.

(ii) Find the partial fractions of $\dfrac{x-2}{(x+1)(x-1)^2}$. Use those partial fractions to evaluate $\displaystyle\int \frac{x-2}{(x+1)(x-1)^2}\,\mathrm{d}x$.

(i) Find the partial fractions of $\dfrac{4x-1}{x^2-4x+4}$. Use those partial fractions to evaluate $\displaystyle\int \frac{4x-1}{x^2-4x+4}\,\mathrm{d}x$

$\left[\text{Answer:}\dfrac{4x-1}{(x-2)^2}=\dfrac{4}{(x-2)}+\dfrac{7}{(x-2)^2}\right].$

(ii) Find the partial fractions of $\dfrac{1}{(x-3)(x+1)^2}$. Use those partial fractions to evaluate $\displaystyle\int \frac{1}{(x-3)(x+1)^2}\,\mathrm{d}x$.

$\left[\text{Answer:}\dfrac{1}{(x-3)(x+1)^2}=\dfrac{1}{16(x-3)}-\dfrac{1}{16(x+1)}-\dfrac{1}{4(x+1)}\right.$

- A quadratic factor is anything of the form $ax^2 + bx + c$, where $a \neq 0$.
- Such a factor is irreducible if the discriminate, $b^2 - 4ac$, is less than 0.
- In the case of irreducible quadratic factors, instead of just putting a letter, we write a polynomial that is one degree less than the denominator.

(i) Find the partial fractions of $\dfrac{(x-1)}{(x+1)(x^2+1)}$. Use those partial fractions to evaluate $\displaystyle\int \frac{(x-1)}{(x+1)(x^2+1)}\,\mathrm{d}x$.

(ii) Find the partial fractions of $\dfrac{x^2-9x+9}{(x^2+1)(x-2)}$. Use those partial fractions to evaluate $\displaystyle\int \frac{x^2-9x+9}{(x^2+1)(x-2)}\,\mathrm{d}x$.

(i) Find the partial fractions of $\dfrac{x^2-9x+9}{(x^2+1)(x-2)}$.

$$\left[\text{Answer:}\frac{x^2-9x+9}{(x^2+1)(x-2)}=\frac{2x-5}{x^2+1}-\frac{1}{x-2}\right].$$

(ii) Find the partial fractions of $\dfrac{x-1}{(x+3)(x^2+3x+2)}$.

$$\left[\text{Answer:}\frac{x-1}{(x+3)(x^2+3x+2)}=\frac{-2}{(x+3)}+\frac{2x+1}{x^2+3x+2}\right].$$

Integration by partial fractions
Improper rational functions

(i) $$\int \frac{x^2 - 2x + 7}{x^2 - 3x + 2}\,\mathrm{d}x$$

(ii) $$\int \frac{x^3 - 2x^2 + 2}{x^2 - 5x + 6}\,\mathrm{d}x$$

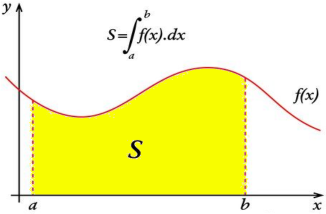
(iii) $$\int \frac{2x - 3}{x^2 - 2x + 1}\,\mathrm{d}x$$

(iv) $$\int \frac{x - 3}{(x-1)(x^2 + x + 4)}\,\mathrm{d}x$$

Definite integrals

Introduction

- A integral of the form $\int_a^b f(x)\,\mathrm{d}x$ is called a **definite integral**.
- The definite integral of $f(x)$ is a **number**.
- It represents the area under the curve $f(x)$ from $x = a$ to $x = b$.



Examples

(i) $\displaystyle\int_{-1}^{1} x \, \mathrm{d}x$

(ii) $\displaystyle\int_{-2}^{2} x^2 \, \mathrm{d}x$

(iii) $\displaystyle\int_{0}^{1} (x^3 + 5) \, \mathrm{d}x$

(iv) $\displaystyle\int_{0}^{\pi/2} \cos 2x \, \mathrm{d}x$

(v) $\displaystyle\int_{-1}^{2} e^{3x} \, \mathrm{d}x$

(vi) $\displaystyle\int_{0}^{1} (5 \sin t + 4t) \, \mathrm{d}t$

Thank You