Geometry

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References

1. Simplified Course in Solid Geometry (S. Chand's).
2. Solid geometry - 1980 -Vikas ; New Delhi - xi,445 p, (516.23ZAM).
3. Analytic geometry - 1973 -Addison-Wesley ; Reading - ix,294 p. (516.3FUL).

Outline

- **Chapter 1** Coordinates and Direction Cosines
- **Chapter 2** The Plane
- **Chapter 3** The Straight Line
- **Chapter 4** The Sphere
- **Chapter 5** The Conicoid

Time allocation

Number of lecture hours = 15

Number of tutorial hours = 8

Chapter 1

Coordinates and Direction Cosines

What is Geometry?

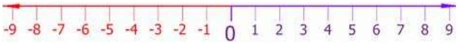
- Geometry is a branch of mathematics concerned with shape, size, relative position of figures, and the properties of space.
- The introduction of coordinates and the concurrent development of algebra marked a new stage for geometry.
- So figures such as plane curves, could now be represented using functions and equations.

Cartesian coordinate system

- Each reference line is called a coordinate axis or just axis of the system, and the point where they meet is its origin.
- They are the most common coordinate system used in computer graphics, computer-aided geometric design, and other geometry-related data processing.
- Cartesian coordinate systems are used in different dimensions.

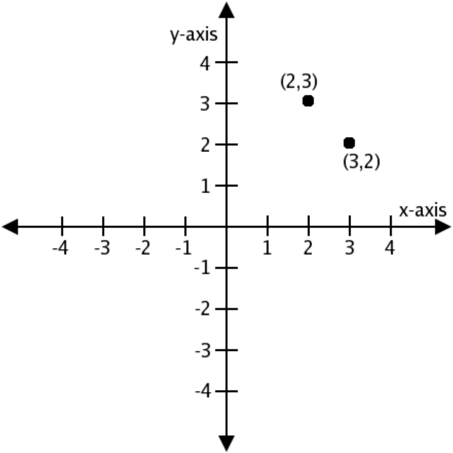
One-dimensional space

In one-dimensional space, choosing a point O of the line (the origin), a unit of length, and an orientation for the line.



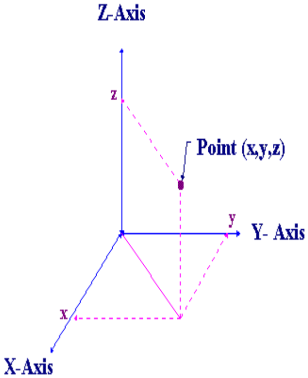
Two dimensional space

In two dimensions (also called a rectangular coordinate system) is defined by an ordered pair of perpendicular lines (axes), a single unit of length for both axes, and an orientation for each axis.

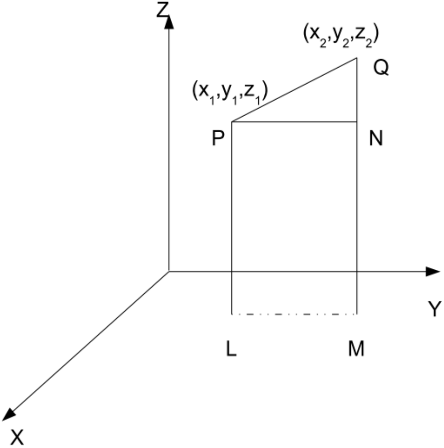


Three-dimensional space

In three-dimensional space means choosing an ordered triplet of lines (axes), any two of them being perpendicular; a single unit of length for all three axes; and an orientation for each axis.



Distance between two points



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Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any points.

Draw PL and QM perpendiculars from P and Q on xy plane.

The coordinates of L and M are (x_1, y_1) and (x_2, y_2).

$$LM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Through P, draw $PN \perp QM$,

Then from the right angled $\triangle\, PNQ$,

Cont...

$$
\begin{aligned}
PQ^2 &= QN^2 + PN^2 \\
&= QN^2 + LM^2 \\
&= (QM - NM)^2 + LM^2 \\
&= LM^2 + (QM - PL)^2 \\
&= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\
PQ^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2
\end{aligned}
$$

Example

Find the values of x if the distances between two points $(x, -8, 4)$ and $(3, -5, 4)$ is 5.

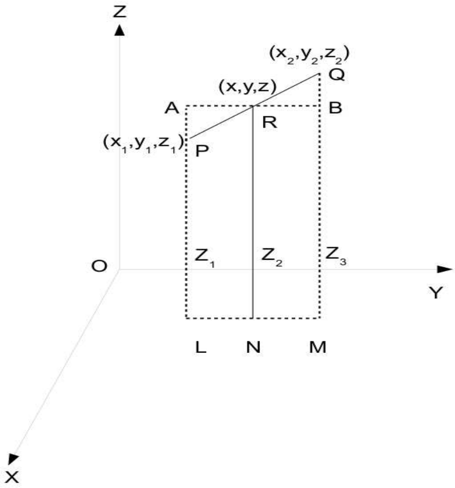
Solution

The given points are $(x, -8, 4)$ and $(3, -5, 4)$.

The distance between these points $=$5.

$$\begin{aligned}
\sqrt{(x-3)^2 + (-8+5)^2 + (4-4)^2} &= 5 \\
(x-3)^2 + 9 + 0 &= 25 \\
(x-3)^2 &= 16 \\
(x-3) &= \pm 4 \\
x &= 7, -1
\end{aligned}$$

Ratio Formula



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The co-ordinates of the point which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ are

$$\left[\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right].$$

Proof

Let $R(x, y, z)$ divides PQ in the ratio $m_1 : m_2$.

Draw PL, QM and RN perpendicular on the xy-plane. Through R, draw a line ARB parallel to LNM and so as to meet LP in A and MQ in B. AB is perpendicular to PL and BM.

PL is parallel to QM.

Hence $PA \parallel BQ$.

The $\triangle$ PAR and $\triangle$ QBR are similar

$$\begin{aligned} \frac{PR}{RQ} &= \frac{PA}{BQ} \\ \frac{m_1}{m_2} &= \frac{z - z_1}{z_2 - z} \end{aligned}$$

Cont...

$$
\begin{aligned}
m_1 z_2 - m_1 z &= m_2 z - m_2 z_1 \\
(m_1 + m_2) z &= m_1 z_2 + m_2 z_1 \\
z &= \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
x &= \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \\
y &= \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}
\end{aligned}
$$

Corollary 1

The coordinates of the mid-point of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right].$$

Corollary 2

The coordinates of the point dividing the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $k : 1$ are

$$\left[\frac{x_1 + kx_2}{k+1}, \frac{y_1 + ky_2}{k+1}, \frac{z_1 + kz_2}{k+1}\right].$$

Example

Find the coordinates of a point which divides the points $(1, 3, 7), (6, 3, 2)$ in the ratio 2:3.

Solution

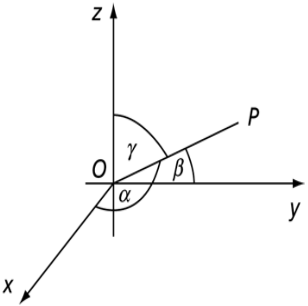
Let the co-ordinates of the required point be (x, y, z).

$$
\begin{aligned}
x &= \frac{2 \times 6 + 3 \times 1}{2+3} = \frac{12+3}{5} = 3 \\
y &= \frac{2 \times 3 + 3 \times 3}{2+3} = \frac{6+9}{5} = 3 \\
z &= \frac{2 \times 2 + 3 \times 7}{2+3} = \frac{4+21}{5} = 5
\end{aligned}
$$

So the required point is (3,3,5).

Direction cosines of a line

If α, β, γ be the angles which any line makes with the positive direction of the axes, then $\cos\alpha, \cos\beta$ and $\cos\gamma$ are called the direction cosines of the given line. The direction cosines are generally denoted as l, m, n.



Relation between direction cosines

If l, m and n are the direction cosines of any line, then

$$\mathbf{l^2 + m^2 + n^2 = 1}.$$

The sum of the squares of the direction cosines of every line is one.

Proof

Let OP be drawn through the origin parallel to the given line so that l, m, n are the cosines of the angles which OP makes with OX, OY, OZ respectively.

Let (x, y, z) be the coordinates of any point P on this line.

Let $OP = r$

$x = lr, y = mr, z = nr.$

Squaring and adding, we obtain

$$\begin{aligned} x^2 + y^2 + z^2 &= (l^2 + m^2 + n^2)r^2 \\ x^2 + y^2 + z^2 &= OP^2 = r^2 \\ l^2 + m^2 + n^2 &= 1. \end{aligned}$$

Example

If α, β, γ be the angles which a line makes with the coordinates axes, then show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Solution

We know that

$$\begin{aligned} l^2 + m^2 + n^2 &= 1 \\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= 1 \\ (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) &= 1 \\ \sin^2\alpha + \sin^2\beta + \sin^2\gamma &= 2 \end{aligned}$$

Direction ratios of a line

Any three numbers a, b, c proportional to the direction cosines of a line are called direction ratios of a line.

Given: Direction ratios a, b, c.
To find: Direction cosines l, m, n.

Solution

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}}$$
$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Direction ratios of a line are 3, 4, 12. What are its direction cosines?

Solution

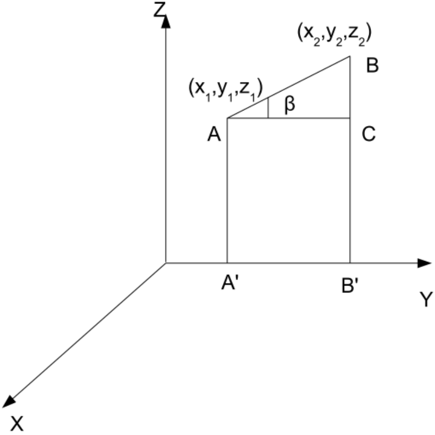
Direction ratios $a = 3, b = 4, c = 12$.

$$l = \frac{3}{\sqrt{3^2 + 4^2 + 12^2}}, m = \frac{4}{\sqrt{3^2 + 4^2 + 12^2}}, n = \frac{12}{\sqrt{3^2 + 4^2 + 12^2}}$$

$$l = \frac{3}{13} \quad m = \frac{4}{13} \quad n = \frac{12}{13}$$

The direction ratios of the line joining two points

$A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be the two given points and let $AB = r$.



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Draw AA' and BB' $\perp$ to OY and $AC \perp BB'$

$$AC = A'B' = OB' - OA' = y_2 - y_1$$

$$\cos\beta = \frac{AC}{AB} = \frac{y_2 - y_1}{r} \text{ or } m = \frac{y_2 - y_1}{r}$$

$$\text{Similarly } l = \frac{x_2 - x_1}{r},\ n = \frac{z_2 - z_1}{r}$$

$$\begin{aligned} l : m : n &= \frac{x_2 - x_1}{r} : \frac{y_2 - y_1}{r} : \frac{z_2 - z_1}{r} \\ &= (x_2 - x_1) : (y_2 - y_1) : (z_2 - z_1) \end{aligned}$$

We get the direction ratios by taking the difference of the corresponding coordinates of two points.

Example

What are the direction cosines of the straight line joining the points $(3, 4, -1)$ and $(1, 7, -1)$?

Solution

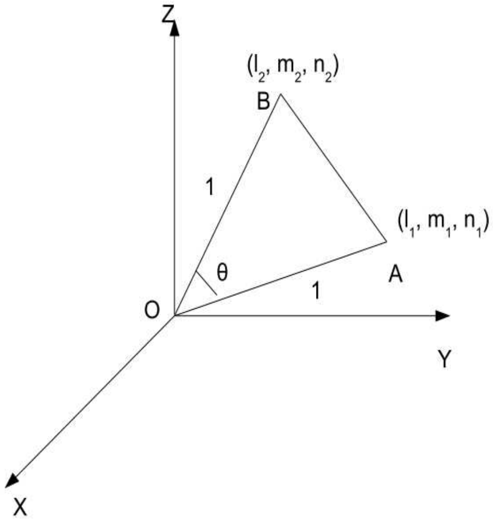
The given points are $(3, 4, -1)$ and $(1, 7, -1)$.
The ratios of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
Here the direction ratios are 1-3, 7-4, -1+1 or -2, 3, 0.
Direction cosines are

$$\frac{-2}{\sqrt{(-2)^2 + (3)^2 + (0)^2}}, \frac{3}{\sqrt{(-2)^2 + (3)^2 + (0)^2}}, \frac{0}{\sqrt{(-2)^2 + (3)^2 + (0)^2}}$$

or

$$\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, 0.$$

Angle between two lines



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Let θ be the angle between the given lines.

Through O, draw OA and OB $\parallel$ to two given lines.

Let $OA = OB = 1$.

The coordinates of A are (l_1, m_1, n_1).

The coordinates of B are (l_2, m_2, n_2).

$$
\begin{aligned}
AB^2 &= (l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2 \\
&= l_1^2 + l_2^2 - 2l_1 l_2 + m_1^2 + m_2^2 - 2m_1 m_2 + n_1^2 + n_2^2 - 2n_1 n_2 \\
&= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) - 2(l_1 l_2 + m_1 m_2 + n_1 n_2) \\
&= 1 + 1 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2) \\
&= 2 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)
\end{aligned}
$$

Cont...

From $\triangle$ AOB, by cosine formula, we have

$$\begin{aligned}
\cos\theta &= \frac{OA^2 + OB^2 - AB^2}{2OA \times OB} \\
\cos\theta &= \frac{1^2 + 1^2 - [2 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)]}{2 \times 1 \times 1} \\
\cos\theta &= \frac{2 - 2 + 2(l_1 l_2 + m_1 m_2 + n_1 n_2)}{2} \\
\cos\theta &= l_1 l_2 + m_1 m_2 + n_1 n_2
\end{aligned}$$

Corollary

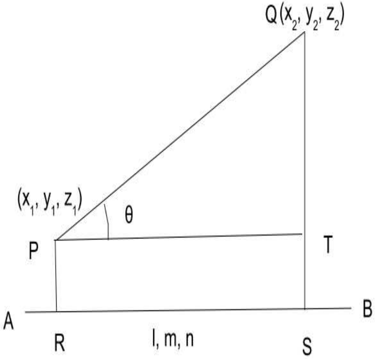
The condition that the lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 are perpendicular is

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Projection of the join of two points

The projection of the join of tow points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line whose direction cosines are l, m, n is

$$\mathbf{l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)}.$$



Proof

Direction ratios of PQ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$.

Its direction cosines are $\dfrac{x_2 - x_1}{r}, \dfrac{y_2 - y_1}{r}, \dfrac{z_2 - z_1}{r}$ where

$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = PQ$.

If θ be the angle between PQ and the given line, then

$$\begin{aligned} \cos\theta &= l.\frac{x_2 - x_1}{r} + m.\frac{y_2 - y_1}{r} + n.\frac{z_2 - z_1}{r} \\ r\cos\theta &= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \end{aligned}$$

Thus the projection of PQ on the given line

$$\begin{aligned} &= RS = PT = PQ\cos\theta = r\cos\theta \\ &= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1) \end{aligned}$$

Hence the result.

Chapter 2

The Plane

What is plane?

- A plane is any flat, two-dimensional surface.
- Many fundamental tasks in geometry, trigonometry, and graphing are performed in the plane.
- Planes can arise as subspaces of some higher dimensional space, as with the walls of a room.

Properties

In three-dimensional Euclidean space, we may exploit the following facts that do not hold in higher dimensions:

- Two planes are either parallel or they intersect in a line.
- A line is either parallel to a plane, intersects it at a single point, or is contained in the plane.
- Two lines perpendicular to the same plane must be parallel to each other.
- Two planes perpendicular to the same line must be parallel to each other.

General equation of the first degree

- Every equation of the first degree in x, y, z represents a plane.
- The most general equation of the first degree in x, y, z is
$$ax + by + cz + d = 0$$
where a, b, c are not all zero.

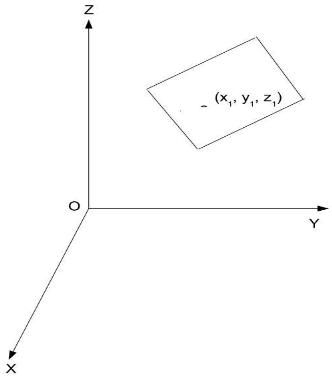
Note

- $x = 0$ is the equation of yz-plane.
- $y = 0$ is the equation of zx-plane.
- $z = 0$ is the equation of xy-plane.

Plane through a given point

The equation of any plane through (x_1, y_1, z_1) is

$$a(x - x_1) + (y - y_1) + c(z - z_1) = 0.$$



Proof

Let the equation of the plane be

$$ax + by + cz + d = 0 \tag{1}$$

As this plane passes through (x_1, y_1, z_1) we have

$$ax_1 + by_1 + cz_1 + d = 0 \tag{2}$$

On subtracting (2) from (1) we eliminate d and get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

Hence the result.

Example

Find the equation of a plane which passes through the point $(2, 3, 4)$.

Solution

Any plane passing through (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

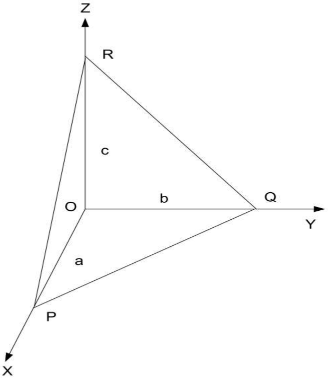
The plane passing through $(2, 3, 4)$ is

$$a(x - 2) + b(y - 3) + c(z - 4) = 0.$$

Intercept form

The equation of the plane in terms of the intercepts of a, b, c from the axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$



Proof

Let the equation of the plane be

$$Ax + By + Cz + D = 0. \tag{1}$$

The plane passes through P, Q, R such that
$OP = a, OQ = b, OR = c$.

The coordinates of P, Q and R are $(a, 0, 0), (0, b, 0), (0, 0, c)$.

As it passes trough $P(a, 0, 0)$

$$\begin{aligned} Aa + D &= 0 \\ A &= -\frac{D}{a} \\ B &= -\frac{D}{b} \\ C &= -\frac{D}{c} \end{aligned}$$

Cont...

Putting the values of A, B and C in (1) we have

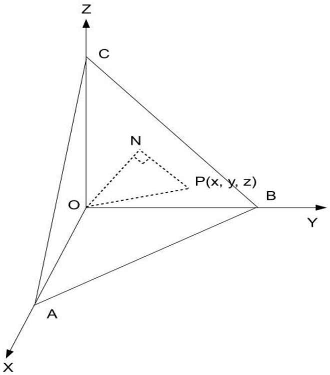
$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Hence the result.

Normal form

The equation of a plane in terms of p, the length of the perpendicular from the origin to it and l, m, n, the direction cosines of that perpendicular is

$$\mathbf{lx} + \mathbf{my} + \mathbf{nz} = \mathbf{p}.$$



Proof

Let ABC be the plane and N the foot of the perpendicular from O on it so that $ON = p$. The direction cosines of ON are l, m, n.

Take any point $P(x, y, z)$ on the plane. Now $PN \perp ON$, since PN lies in the plane which is $\perp$ to ON.

Therefore, the projection of OP on ON,

$$= ON = p. \tag{1}$$

Also the projection of the line OP joining $O(0,0,0)$ and $P(x, y, z)$, on the line ON whose direction cosines are l, m, n is

$$l(x-0) + m(y-0) + n(z-0). \tag{2}$$

From (1) and (2) we have

$$lx + my + nz = p$$

Corollary

The general equation of a plane can be reduced to normal form as

$$ax + by + cz + d = 0$$

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z + \frac{d}{\sqrt{a^2+b^2+c^2}} = 0$$

Direction cosines of normal to a plane

The direction ratios of normal to any plane are the coefficients of x, y, z in its equation.

Thus a, b, c are the direction ratios of the normal to the plane

$$ax + by + cz + d = 0.$$

Example

Find the direction cosines of the normal to the plane $3x + 4y + 12z = 52$. Also find the length of the perpendicular from the origin to the plane.

Solution

$$3x + 4y + 12z = 52$$

Dividing by $\sqrt{3^2 + 4^2 + 12^2}$ i.e by 13

$$\frac{3}{13}x + \frac{4}{13}y + \frac{12}{13}z = 4$$

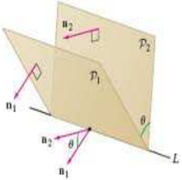
The direction cosines of the normal are $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.

Length of the perpendicular from origin $=4$.

Angle between two planes

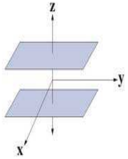
Angle between two planes is equal to the angle between the normals to them from any point. Thus, the angle between the two planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ is equal to the angles between the lines with direction ratios a, b, c and a_1, b_1, c_1 and is, therefore

$$= \cos^{-1}\left[\frac{aa_1 + bb_1 + cc_1}{\sqrt{a^2 + b^2 + c^2}\sqrt{a_1^2 + b_1^2 + c_1^2}}\right].$$



Parallelism and perpendicularity of two planes

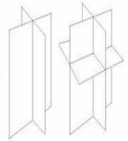
Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular. Thus the two planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ will be parallel, if



$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$$

and will be perpendicular, if

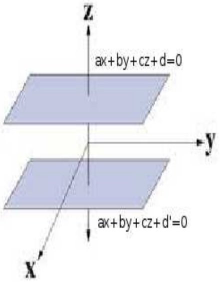
$$aa_1 + bb_1 + cc_1 = 0.$$



Corollary

The equation of a plane parallel to the plane $ax + by + cz + d = 0$ is

$$ax + by + cz + d' = 0.$$



Example

Find the equation of the plane through $(0, 1, -2)$ parallel to the plane $2x - 3y + 4z = 0$.

Solution The equation of the plane parallel to the given plane is $2x - 3y + 4z = d$.
Since above plane passing through $(0, 1, -2)$, we have

$$\begin{aligned} 2(0) - 3(1) + 4(-2) &= d \\ -3 - 8 &= d \\ d &= -11 \end{aligned}$$

On substituting the value of d in above equation, we get
$2x - 3y + 4z = -11$

Plane through three points

The equation of the plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Example 1

Find the equation of the plane through the points $(2, 2, 1)$, $(1, -2, 3)$ and parallel to the x-axis.

Example 2

Find the equation of the plane passing through the intersection of the planes $2x + y + 2z = 9, 4x - 5y - 4z = 1$ and the point $(3, 2, 1)$.

Solution

$$2x + y + 2z = 9 \tag{1}$$
$$4x - 5y - 4z = 1 \tag{2}$$

The equation of a plane through the line of intersection of the planes (1) and (2) is

$$2x + y + 2z - 9 + k(4x - 5y - 4z - 1) = 0 \tag{3}$$

The plane (3) passes through $(3, 2, 1)$, then

$$\begin{aligned} 2(3) + 2 + 2(1) - 9 + k(4(3) - 5(2) - 4(1) - 1) &= 0 && (4)\\ K &= \frac{1}{3} && (5) \end{aligned}$$

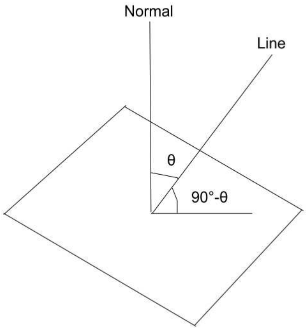
Cont...

On substituting the values of k in (3), we get

$$\begin{aligned} 2x + y + 2z - 9 + \frac{1}{3}(4x - 5y - 4z - 1) &= 0 \\ 10x - 2y + 2z &= 28 \end{aligned}$$

Angle between a plane and a line

- It is the complement of the angle between the normal to the plane and the line.
- If θ is the angle between the normal to th
  90°- θ is the angle between the plane anc



Example

Find the angle between the plane $x + 2y - 3z + 4 = 0$ and the line whose direction cosines are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Solution

The direction ratios of the normal to $x + 2y - 3z + 4 = 0$ are $1, 2, -3$.

The direction cosines of the normal are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}$.

The direction cosines of the line are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Let θ be the angle between the normal and the given line.

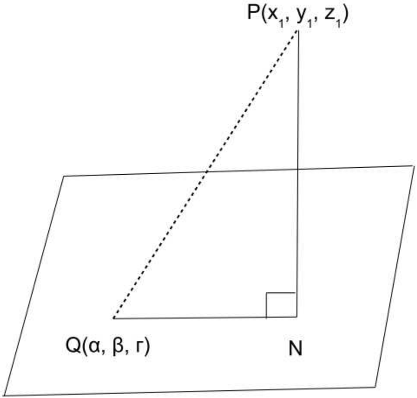
Cont...

Then

$$
\begin{aligned}
\cos\theta &= \frac{1}{\sqrt{14}}.\frac{2}{\sqrt{14}} + \frac{2}{\sqrt{14}}.\frac{3}{\sqrt{14}} - \frac{3}{\sqrt{14}}.\frac{1}{\sqrt{14}} \\
&= \frac{2}{14} + \frac{6}{14} - \frac{3}{14} \\
&= \frac{5}{14} \\
\theta &= \cos^{-1}\left[\frac{5}{14}\right]
\end{aligned}
$$

90°-θ is the angle between the plane and the line.

Length of a perpendicular from a point to a plane



Proof

Let PN be the length of the perpendicular from $P(x_1, y_1, z_1)$ to the given plane

$$ax + by + cz + d = 0 \tag{1}$$

PN is the normal to (1).
Therefore its direction cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Let $Q(\alpha, \beta, \gamma)$ be a point on (1), so that

$$a\alpha + b\beta + c\gamma + d = 0 \tag{2}$$

PN=projection of PQ on PN

$$= \frac{a(x_1-\alpha)}{\sqrt{a^2+b^2+c^2}} + \frac{b(y_1-\beta)}{\sqrt{a^2+b^2+c^2}} + \frac{c(z_1-\gamma)}{\sqrt{a^2+b^2+c^2}}$$

Cont...

$$= \frac{ax_1 + by_1 + cz_1 - (a\alpha + b\beta + c\gamma)}{\sqrt{a^2 + b^2 + c^2}}$$

[From (2), $a\alpha + b\beta + c\gamma = -d$]

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Example

Find the length of the perpendicular from the origin to the plane $2x + 3y + 4z + 5 = 0$.

Solution Length of the perpendicular from (x_1, y_1, z_1) to plane is

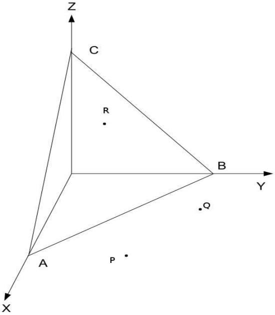
$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}$$

Length of the perpendicular from $(0, 0, 0)$ to plane is

$$\begin{aligned} &= \frac{2(0) + 3(0) + 4(0) + 5}{\sqrt{2^2 + 3^2 + 4^2}} \\ &= \frac{5}{\sqrt{29}} \end{aligned}$$

Corollary 1

The two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or one opposite side of the plane $ax + by + cz + d = 0$ is determined as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same sign or of opposite sign.



Example

Find the distance of the points $(3, 2, 6)$ and $(1, 1, 0)$ from the plane $2x + 2y + z - 10 = 0$?
Are these points on the same side of the plane?

Solution

The equation of the plane is $2x + 2y + z - 10 = 0$.
For point $(3, 2, 6) \Rrightarrow$

$$\text{Length of perpendicular} = \frac{2(3) + 2(2) + (6) - 10}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{6}{3} = +2$$

For point $(1, 1, 0) \Rrightarrow$

$$\text{Length of perpendicular} = \frac{2(1) + 2(1) + (0) - 10}{\sqrt{2^2 + 2^2 + 1^2}} = -\frac{6}{3} = -2$$

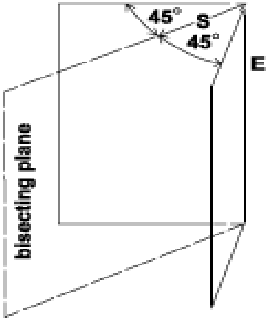
Since the result of substituting the coordinates of the two points in the equation of the plane gives opposite signs, the two points lie on the opposite sides of the plane.

Planes bisecting the angle

Let the equation of the two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$a_2x + b_2y + c_2z + d_2 = 0. \tag{2}$$



Cont...

Let $P(x, y, z)$ be any point on the bisecting plane. Then perpendicular distance of P from each of the given planes will be the same.

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \qquad (3)$$

Let θ be the angle between (1) and (2).
If $\theta < 45^\circ$, then (3) bisects the acute angle.

Example

Find the plane which bisect the acute angle between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$.

Solution

$$x + 2y + 2z = 9 \tag{1}$$
$$4x - 3y + 12z + 13 = 0 \tag{2}$$

The equation of the bisecting planes are

$$\frac{x + 2y + 2z - 9}{\sqrt{1 + 4 + 4}} = \pm \frac{4x - 3y + 12z + 13}{\sqrt{16 + 9 + 144}}$$

Cont...

$$\begin{aligned} 13(x+2y+2z-9) &= \pm 3(4x-3y+12z+13) \\ 13x+26y+26z-117 &= \pm(12x-9y+36z+39) \end{aligned}$$

$$x+35y-10z-156=0 \quad (3)$$

$$25x+17y+62z-78=0 \quad (4)$$

Let θ be the angle between the plane (1) and plane (4)

$$\begin{aligned} \cos\theta &= l_1 l_2 + m_1 m_2 + n_1 n_2 \\ &= \left[\frac{1}{3}\right]\left[\frac{25}{\sqrt{4758}}\right] + \left[\frac{2}{3}\right]\left[\frac{17}{\sqrt{4758}}\right] + \left[\frac{2}{3}\right]\left[\frac{62}{\sqrt{4758}}\right] \\ &= .8843 \end{aligned}$$

$$\Rrightarrow \theta < 45\ ^\circ$$

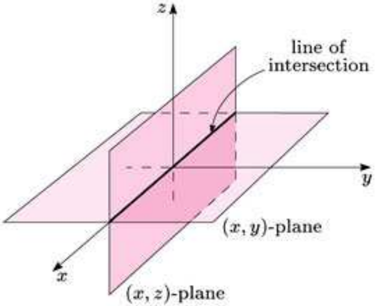
Hence (4) is the bisector of the acute angle between the given planes.

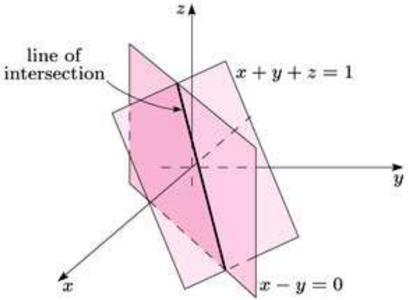
Chapter 3

The Straight Line

Line

- A line may be determined as the intersection of any two planes.
- The intersection of two planes ZOY, ZOX gives the equation of z-axis.





The equation of general form of straight line

The general form of equation of straight line is

$$ax + by + cz + d = 0 \tag{1}$$

$$a_1x + b_1y + c_1z + d_1 = 0. \tag{2}$$

Symmetrical form

- The equation of the line passing through a given point $A(x_1, y_1, z_1)$ and having direction cosines l, m, n are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

- The equation of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

as the direction ratios of the line are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Example 1

Find the equation of the line passing through the points $(2, 3, 4)$ and $(4, 6, 5)$.

Solution The line passing through $(2, 3, 4)$ and $(4, 6, 5)$.

The direction ratios of the line are $4 - 2, 6 - 3, 5 - 4$.

The equation of the line are

$$\frac{x-2}{4-2} = \frac{y-3}{6-3} = \frac{z-4}{5-4}$$
$$\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-4}{1}$$

Example 2

Find the symmetrical form of equations of the line of intersection of the planes $x + y + z + 1 = 0, 4x + y - 2z + 2 = 0$.

Solution

$$x + y + z + 1 = 0 \tag{1}$$
$$4x + y - 2z + 2 = 0 \tag{2}$$

Let $z = 0$, then

$$x + y = -1 \tag{3}$$
$$4x + y = -2 \tag{4}$$

By solving (3) and (4). we have $x = -\frac{1}{3}, -\frac{2}{3}$.

Cont...

A point which lies on the line is $\left[x = -\frac{1}{3}, -\frac{2}{3}, 0\right]$.

let l, m, n be the direction ratios of the line, therefore

$$l + m + n = 0 \tag{5}$$

$$4l + m - 2n = 0 \tag{6}$$

It implies that

$$\frac{l}{-2-1} = \frac{m}{4+2} = \frac{n}{1-4}$$

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

Hence the equation of a line is

$$\frac{x + \frac{1}{3}}{1} = \frac{y + \frac{2}{3}}{-2} = \frac{z - 0}{1}$$

Angle between a line and a plane

Angle between a line and plane is the complement of the angle between the line and the normal to the plane,

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \tag{1}$$

$$ax+by+cz+d=0 \tag{2}$$

Let θ be the angle between the plane and the line, then 90°-θ is the angle between normal and the line.

$$\begin{aligned}\cos(90^\circ-\theta) &= \frac{aa_1+bb_1+cc_1}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}\\ \sin\theta &= \frac{aa_1+bb_1+cc_1}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}\end{aligned}$$

Corollary

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \qquad \text{(line)}$$
$$ax + by + cz + d = 0 \qquad \text{(plane)}$$

The required condition for straight line to be paralleled to the plane is

$$\mathbf{aa_1 + bb_1 + cc_1 = 0}.$$

Condition for a line to lie in a plane

A line will be lie in a given plane if

- the normal to the plane is perpendicular to the line,
- any one point on the line lies in the plane.

Example

Find the equation of the plane through $(0, 2, 4)$ and containing the line

$$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}.$$

Solution Equation of a plane passing through $(0, 2, 4)$ is

$$a(x-0) + b(y-2) + c(z-4) = 0 \tag{1}$$

If plane (1) contains the given line then

$$3a + 4b - 2c = 0 \tag{2}$$

If point (-3, 1, 2) of a given line lies on (1) then

$$a(-3-0) + b(1-2) + c(2-4) = 0 \text{ or } -3a - b - 2c = 0 \tag{3}$$

Cont...

On solving (2) and (3) we get

$$\frac{a}{-8-2} = \frac{b}{6+6} = \frac{c}{-3+12}.$$

or

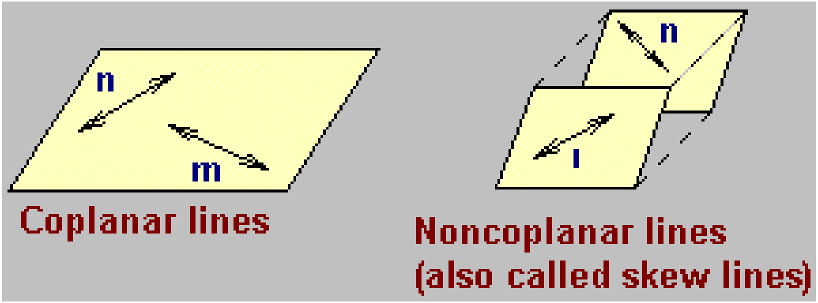
$$\frac{a}{10} = \frac{b}{-12} = \frac{c}{-9}.$$

On substituting the values of a, b, and c in (1) we get

$$\begin{aligned} 10(x-0) - 12(y-2) - 9(z-4) &= 0 \\ 10x - 12y - 9z + 60 &= 0. \end{aligned}$$

Coplanar lines

- Coplanar lines are lines lying on the same plane.
- Lines in the same plane are not always parallel.
- Parallel lines in the same plane will not intersect.
- Two lines which are not coplanar are called skew lines.



Example

Prove that the straight lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar.

Solution

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \tag{1}$$

$$4x - 3y + 1 = 0 = 5x - 3z + 2 \tag{2}$$

The equation of the plane through (2) is

$$4x - 3y + 1 + k(5x - 3z + 2) = 0$$

$$(4 + 5k)x + (-3)y + (-3k)z + 2k + 1 = 0 \tag{3}$$

Cont...

This plane (3) will be parallel to line (1), if

$$\begin{aligned} 2(4+5k)+3(-3)+4(-3k) &= 0 \\ k &= -\frac{1}{2} \end{aligned}$$

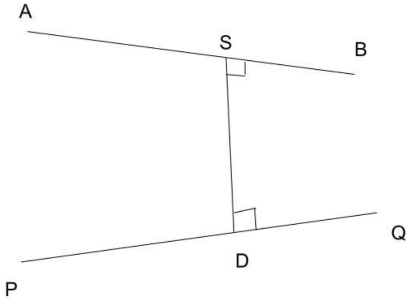
Putting the values of k in (3), we have

$$\begin{aligned} \left[4-\frac{5}{2}\right]x-3y-3\left[-\frac{1}{2}\right]z+2\left[-\frac{1}{2}\right]+1=0 \\ \frac{3}{2}x-3y+\frac{3}{2}z-1+1=0 \\ x-2y+z=0 \end{aligned}$$

This is the equation of the plane containing line (2) and parallel to (1).
A point (1, 2, 3) of line (1) also lies on it [1-2(2)+3=0].
Plane (4) contains both lines (1) and (2).

The shortest distance between two lines

Let AB and PQ be the skew lines. Let a line SD be a perpendicular to AB and PQ. Then the length of line SD is called the shortest distance between them.



Chapter 4

The Sphere

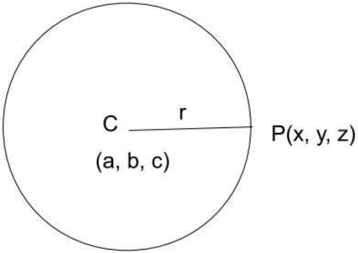
Definition

- A sphere is the locus of a point which remains at a constant distance from a fixed point.
- The constant distance is called the radius and the fixed point the center of the sphere.

The equation of the sphere

The equation of the sphere whose center is (a, b, c) and radius r, is

$$(\mathbf{x}-\mathbf{a})^2+(\mathbf{y}-\mathbf{b})^2+(\mathbf{z}-\mathbf{c})^2=\mathbf{r}^2.$$



Characteristics of the equation of the sphere

- It is of the second degree in x, y, z.
- The coefficient of x^2, y^2, z^2 are all equal.
- The product terms xy, yz, zx are absent.

The general equation

The equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

represents a sphere whose center is $(-u, -v, -w)$ and radius $= \sqrt{u^2 + v^2 + w^2 - d}$.

Example

$\lambda x^2 + 5y^2 + \mu z^2 - axy + 2\lambda x - 4\mu y - 2az + d = 0$ represents a sphere of radius 2, find the value of d.

Solution

$$\lambda x^2 + 5y^2 + \mu z^2 - axy + 2\lambda x - 4\mu y - 2az + d = 0 \qquad (1)$$

As (1) represents a sphere , therefore the coefficient of x^2, y^2, z^2 should be equal, $\lambda = 5 = \mu$ and coefficient of $xy = 0$, $\Rrightarrow a = 0$

$$\begin{aligned}
\text{Radius} &= \sqrt{u^2 + v^2 + w^2 - d} \\
&= \sqrt{\lambda^2 + (-2\mu)^2 + (-a)^2 - d} \\
2 &= \sqrt{(5)^2 + (-10)^2 + (0)^2 - d} \\
4 &= 125 - d \\
d &= 121
\end{aligned}$$

The sphere through four points

The general equation of a sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$$

It contains four constant u, v, w and d. The sphere passes through four points, each of which gives one independent equation in the constant. We can find the values of four constant u, v, w, d by solving the four equations.

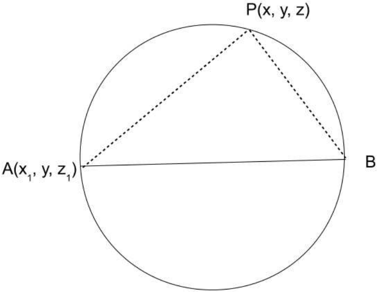
Example

Find the equation of the sphere passing through the four points $(0,0,0),(a,0,0),(0,b,0)$ and $(0,0,c)$.

Diametrical form

The equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is given by

$$\mathbf{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0}.$$



Example

Write down the equation of a sphere one of whom diameter has the end points $(2, -1, 3)$ and $(0, 4, -5)$. Find its center.

Solution

Equation of a sphere described on the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0.$$

Equation of the sphere described on the line joining the points $(2, -1, 3)$ and $(0, 4, -5)$ as diameter is

$$\begin{aligned} (x - 2)(x - 0) + (y + 1)(y - 4) + (z - 3)(z + 5) &= 0 \\ x^2 + y^2 + z^2 - 2x - 3y + 2z - 19 &= 0. \end{aligned}$$

Cont...

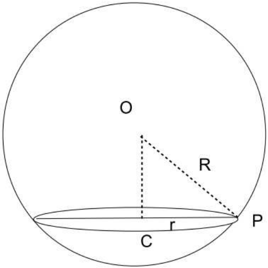
Co-ordinates of the center are

$$\begin{aligned} &= \left[-\frac{1}{2}(\text{coeff of x}), -\frac{1}{2}(\text{coeff of y}), -\frac{1}{2}(\text{coeff of z})\right] \\ &= \left[-\frac{1}{2}(-2), -\frac{1}{2}(-3), -\frac{1}{2}(2)\right] \\ &= \left[1, \frac{3}{2}, -1\right] \end{aligned}$$

Plane section of a sphere is a circle

Two equation, one of a sphere and the other of a plane, together represents a circle

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{sphere}$$
$$ax + by + cz + d_1 = 0. \qquad \text{plane}$$



Cont...

- The center of the circle is the foot of the perpendicular from the center of the sphere on the plane of the circle.
- Radius of a circle $= \sqrt{R^2 - P^2}$ where P is the length of the perpendicular from the center of the sphere to the plane.

Example

Find the center and radius of the circle
$x^2 + y^2 + z^2 - 2x + 4y + 2z - 6 = 0; x + 2y + 2z - 4 = 0.$

Intersection of two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad (1)$$
$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \qquad (2)$$

The curve of intersection of two spheres is a circle.

The sphere through the intersection of two spheres

The equation of any sphere through the intersection of two spheres

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad (1)$$
$$x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \qquad (2)$$

is given by

$$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1$$
$$+k(x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2) = 0.$$

Example

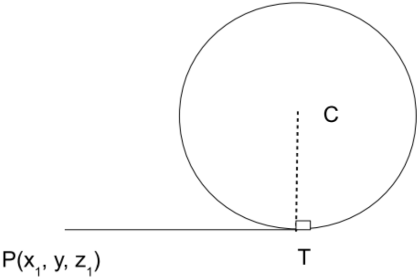
Obtain the equations to the sphere through the common circle of the sphere $x^2 + y^2 + z^2 + 2x + 2y = 0$ and the plane $x + y + z + 4 = 0$ and which intersects the plane $x + y = 0$ in a circle of radius 3 units.

Length of tangent

The length of the tangent drawn from (x_1, y_1, z_1) to a sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $\mathbf{x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d}$.



Example 1

Prove that the length of a tangent drawn from the point $(1, -1, 0)$ to the sphere $x^2 + y^2 + z^2 - 2x + 2y - 2z + 11 = 0$ is 3.

Solution

$$x^2 + y^2 + z^2 - 2x + 2y - 2z + 11 = 0 \tag{1}$$

Length of the tangent from the point $(1, -1, 0)$ to the sphere (1) is

$$\begin{aligned} &= \sqrt{(1)^2 + (-1)^2 + (0)^2 - 2(1) + 2(-1) - 2(0) + 11} \\ &= 3 \end{aligned}$$

Example 2

Find the co-ordinates of the points where the line

$$\frac{1}{4}(x+3)=\frac{1}{3}(y+4)=-\frac{1}{5}(z-8)$$

intersects the sphere $x^2+y^2+z^2+2x-10y=23$.

Cont...

Solution

$$x^2 + y^2 + z^2 + 2x - 10y = 23 \quad (1)$$

$$\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = r \quad (2)$$

Any point on the line (1) is $(4r-3, 3r-4, -5r+8)$.

If this point lies on (1), then

$$\begin{aligned}
(4r-3)^2 + (3r-4)^2 + (-5r+8)^2 + 2(4r-3) - 10(3r-4) &= 23 \\
50r^2 - 150r + 100 &= 0 \\
r^2 - 3r + 2 &= 0 \\
(r-1)(r-2) &= 0
\end{aligned}$$

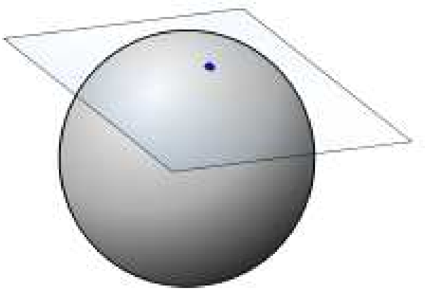
So $r = 1, 2$.

The point of intersection are $(1, -1, 3)$ and $(5, 2, -2)$.

The equation of the tangent plane

The tangent plane at $T(x_1, y_1, z_1)$ on the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is

$$\mathbf{xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0}.$$



Example

Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z - 6 = 0$ at $(1, 2, 3)$.

Solution

$$x^2 + y^2 + z^2 + 2x + 4y - 6z - 6 = 0 \tag{1}$$

Equation of the tangent plane to (1) at $(1, 2, 3)$ is

$$\begin{aligned} xx_1 + yy_1 + zz_1 + 1(x + x_1) + 2(y + y_1) - 3(z + z_1) - 6 &= 0 \\ 1x + 2y + 3z + (x + 1) + 2(y + 20 - 3(z + 3) - 6 &= 0 \\ x + 2y &= 5. \end{aligned}$$

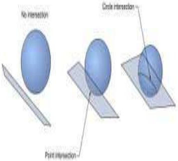
Condition of tangency

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{(sphere)}$$
$$ax + by + cz + d_1 = 0 \qquad \text{(plane)}$$

A plane will touch the sphere if the length of perpendicular from the center $(-u, -v, -w)$ of sphere to plane $=$ Radius of the sphere.

Therefore the required condition for tangency is,

$$\frac{-au - bv - cw + d_1}{\sqrt{a^2 + b^2 + c^2}} = \sqrt{u^2 + v^2 + w^2 - d}.$$



Chapter 5

The Conicoid

Conicoid

- What will a general second degree equation in three variables represent?
- In this unit we study the most general form of a second degree equation in three variables. The surface generated by these equations are called quadrics or conicoids.
- We have studied some particular forms of second degree equations in three variables, namely, those representing spheres, cones.

Cont...

The surface represented by the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

is called a Conicoid or Quadric.

- **case 1** : Suppose we put $a = b = c = 1$ and $g = h = f = 0$ in (1). Then we get the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{(sphere)}$$

- **case 2** : Suppose we put $u = v = w = d = 0$ in (1) , then we get

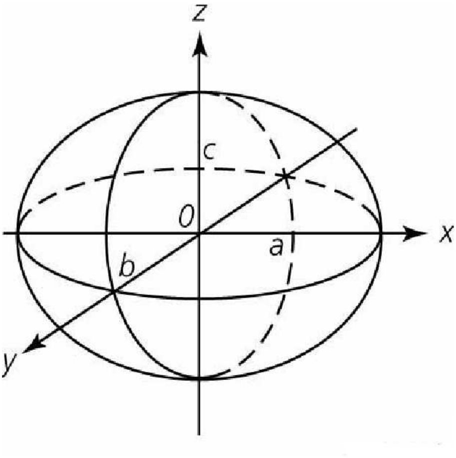
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \qquad \text{(cone)}$$

Cont...

The general equation of a second degree by transformation of co-ordinates axes can be reduced to any one of the following form

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$	Imaginary Ellipsoid
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperboloid of one sheet
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperboloid of two sheet
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$	Imaginary cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	Cone

The Ellipsoid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (1)$$

Cont...

- **Center** Since (x, y, z) and $(-x, -y, -z)$ satisfy (1), it implies that origin bisects every chord which passes through it.
- **Symmetry** Surfaces is symmetrical about the XOY, YOZ and ZOX planes. These three plane are called *Principal Planes*. The three lines of intersection of these planes when in pairs are called *Principal axes*.

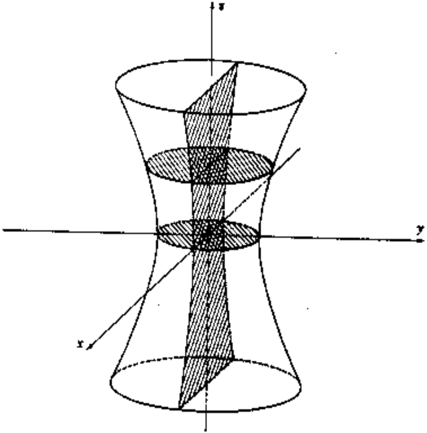
Cont...

- **Closed Surface** The surface is closed.
- **Length of axes** The length of axes of the ellipsoid are $2a, 2b, 2c$.
- **Section of the surface** The plane $z = k$ parallel to XOY plane makes the section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2} \tag{2}$$

where $-c \leq k \leq$. These equation represent ellipses. So an ellipsoid is generated by the ellipse (2).

The Hyperbloid of one sheet



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (1)$$

Cont...

- **Center** The origin bisects all the chords through it, therefore origin is the center of (1).
- **Principal planes** The co-ordinates planes bisect all chords perpendicular to them so called **Principal planes**.
- **Length of axes** The x-axis intersects the surface (1) at $(a, 0, 0)$ and $(-a, 0, 0)$, so that intercept length $2a$ on x-axis. Similarly the length intercepted on y-axis is $2b$. The z-axis does not meet the surface (1).

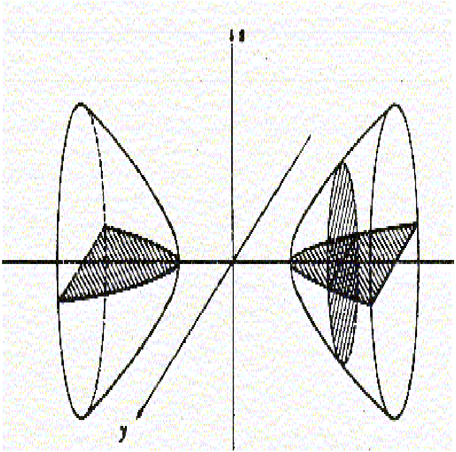
Cont...

- **Sections** A section by planes $z = k$ parallel to XOY plane are ellipses.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, z = k \tag{2}$$

The center of these ellipses lie on z-axis. Here k varies from $-\infty$ to $+\infty$.

The Hyperbloid of two sheets



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (1)$$

Cont...

- **Center** Origin is the center. Co-ordinates planes are the principal planes. Co-ordinates axes are the principal axes of the surface.
- **Points of intersection** X-axis meets the surface in the points $(a, 0, 0)$ and $(-a, 0, 0)$, where as y-axis and z-axis do not meet the surface.

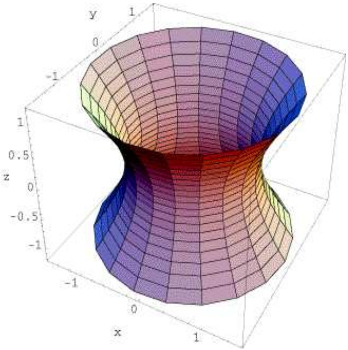
Cont...

- **Sections** The sections by the planes $z = k$ and $y = k$ and the hyperbolas.

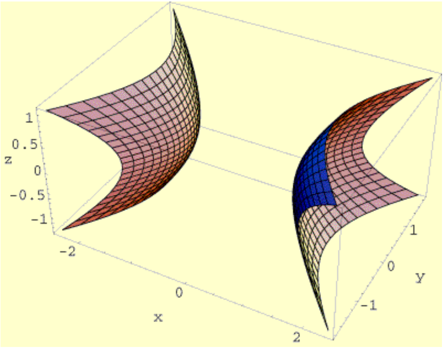
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2} \text{ and } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}. \tag{2}$$

- **Region** The plane $x = k$ does not meet the surface as $-a < k < a$. So there is no portion of surface lie between $x = -a$ and $x = a$.

Computer generated output



(a) Hyperbloid of one sheet



(b) Hyperbloid of two sheets

Figure: Computer generated output.

Central Conicoids

Consider the equation

$$ax^2 + by^2 + cz^2 = 1$$

- If a, b, c are positive, the surface is an ellipsoid.
- If two are positive and one is negative, hyperboloid of one sheet.
- If two are negative and one is positive, hyperboloid of two sheets.

Remark All these surfaces have a center and three principal planes and are as such known as *Central conicoid*.

Intersection of a line with a conicoid

To find the point of intersection of the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \tag{1}$$

with the central conicoid

$$ax^2 + by^2 + cz^2 = 1. \tag{2}$$

If any point $(lr+\alpha, mr+\beta, nr+\gamma)$ of (1) lie on conicoid (2), then

$$a(lr+\alpha)^2 + b(mr+\beta)^2 + c(nr+\gamma)^2 = 1$$

Cont...

$$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \qquad (3*)$$

Equation (3) is a quadratic equation in r. There will be its two roots. If roots are real, then the line will intersect the conicoid at two points.

Tangent plane to a conicoid

Let

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \tag{1}$$

be a line.

$$ax^2 + by^2 + cz^2 = 1 \tag{2}$$

be a conicoid.
If (α, β, γ) lies on (2) then

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \tag{3}$$

Cont...

If line (1) will touch the conicoid (2) then both the roots of r from the equation (3*) are zero. So

$$al\alpha + bm\beta + cn\gamma = 0 \tag{4}$$

This is the condition that line (1) will touch the conicoid (2).

Tangent plane is the locus of the tangent line and its equation is obtained by eliminating l, m, n between (1) and (4).

$$\begin{aligned} a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) &= 0 \\ a\alpha x + b\beta y + c\gamma z &= a\alpha^2 + b\beta^2 + c\gamma^2 \\ a\alpha x + b\beta y + c\gamma &= 1 \end{aligned}$$

The equation of tangent plane is $a\alpha x + b\beta y + c\gamma = 1$.

Condition of tangency

The equation of the plane is

$$lx + my + nz = p. \tag{1}$$

The equation of the central conicoid is

$$ax^2 + by^2 + cz^2 = 1. \tag{2}$$

Let (α, β, γ) be the point of contact at which plane (1) touches the conicoid (2).

The equation of tangent plane is

$$a\alpha x + b\beta y + c\gamma = 1 \qquad (3$$

Cont...

Plane (1) and (3) represent the tangent plane.

Comparing the two equations (1) and (3), we get

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p}$$

$$\alpha = \frac{l}{ap},\ \beta = \frac{m}{bp},\ \gamma = \frac{n}{cp}$$

Since (α, β, γ) lies on (2), therefore

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \tag{4}$$

Putting the values of α, β, γ in (4), we obtain the condition

Cont...

$$\begin{aligned}\frac{al^2}{a^2p^2}+\frac{bm^2}{b^2m^2}+\frac{cn^2}{c^2p^2} &= 1\\ \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c} &= p^2\end{aligned}$$

Putting the values of P in (1) we get the required equation of the tangent plane

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

The point of contact is $\left[\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right]$.

Normal to a conicoid

Let

$$ax^2 + by^2 + cz^2 = 1 \tag{1}$$

be the equation of a central conicoid. Then the equation of a normal is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} = r \text{ (say)}. \tag{2}$$

If (2) passes through a point (f, g, h), then

$$\frac{f-\alpha}{a\alpha} = \frac{g-\beta}{b\beta} = \frac{h-\gamma}{c\gamma} = r \text{ (say)}. \tag{6}$$

Cont...

$$\alpha = \frac{f}{1 + ar}, \ \beta = \frac{g}{1 + br}, \ \gamma = \frac{h}{1 + cr} \tag{3}$$

Since (α, β, γ) lies on the conicoid (1), we have the relation

$$\frac{af^2}{(1 + ar)^2} + \frac{bg^2}{(1 + br)^2} + \frac{ch^2}{(1 + cr)^2} = 1, \tag{4}$$

which, being as equation of the sixth degree, gives six values of r, to each of which there corresponding a point (α, β, γ), obtained from (4).

N.B *Through a given point, six normals, in general can be drawn to a central quadric.*

Thanks