Calculus (Real Analysis I)

(MAT122β)

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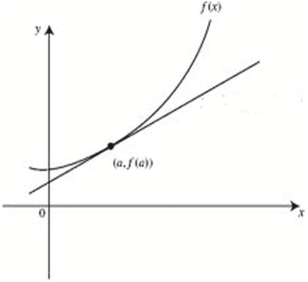
Differentiability

What is differentiability?

- Differentiability arises from the geometric concept of the tangent to a graph.

- We say that f is differentiable at a if the graph $y = f(x)$ has a tangent at the point $(a, f(a))$.

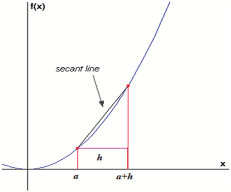


Definition
Defferentiability

Let f be defined on an open interval I, and $a \in I$. Then the derivative of f at a is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

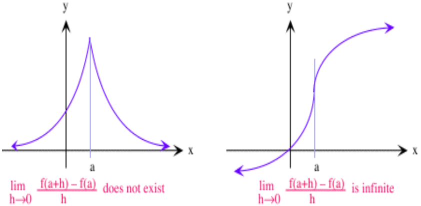
provided that this limit exists. In this case, we say that f is **differentiable** at a.



Non-differentiability

A function can fail to be differentiable at a point a if either

- $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, does not exist,
- or is infinite.



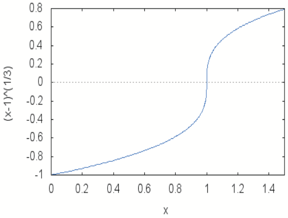
Example 1

Check the differentiability of the function $f(x) = (x-1)^{\frac{1}{3}}$ at $x = 1$.

Example 1
Solution

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\
&= \lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h}\\
&= \lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}\to +\infty.
\end{aligned}
$$

So, f is not differentiable at $x = 1$.



Example 2

Determine points of non-differentiability of the function $f(x) = |x + 2|$.

Example 2
Solution

$$f(x) = |x+2| = \begin{cases} -(x+2) & \text{if } x \leq -2 \\ (x+2) & \text{if } x > -2. \end{cases}$$

Since we know that both $-(x+2)$ and $x+2$ are differentiable, the only point where something can go wrong is when $x = -2$. At this point, we can compute the limit of the difference quotient directly:

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \to 0} \frac{|h|}{h} \end{aligned}$$

However, this limit does not exist since the left and right limits differ.

Hence, f is not differentiable at $x = -2$

Definition ($\epsilon - \delta$)
Differentiability

Let f be defined on an open interval I, and $a \in I$. Then f is **differentiable** at a with derivative $f'(a)$ if:

for each positive number ϵ, there is a positive number δ such that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \epsilon,$$

for all x satisfying $0 < |x - a| < \delta$.

Theorem 5.1

Differentiability and continuity

Let f be defined on an open interval I, and $a \in I$. If f is differentiable at a, then f is also continuous at a.

Theorem 5.1

Proof

If f is differentiable at a, then there is some number $f'(a)$ such that

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=f'(a).$$

It follows that

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\times h\right)\\ &= \left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\times\lim_{h\to 0}(h)\right)\\ &= f'(a)\times 0\\ &= 0.\end{aligned}$$

Hence, by the Sum and Multiple Rules for limits

$$\lim_{h\to 0}f(a+h)=f(a).$$

Thus f is continuous at a.

Corollary 1

If f is discontinuous at a, then f is not differentiable at a.

Example

Check the differentiability of the following function at $x = 0$:

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

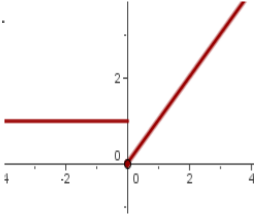
Example
Solution

First, we check the continuity of f at $x = 0$.

$$\lim_{x \to 0^-} 1 = 1, \qquad (1)$$
$$\lim_{x \to 0^+} x = 0. \qquad (2)$$

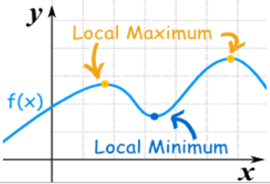
Since (1) $\neq$ (2), f is not continuous at $x = 0$.

It implies that f cannot be differentiable at $x = 0$.



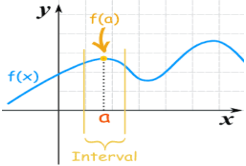
Local maximum and local minimum

- The **local maximum** and **local minimum** (plural: maxima and minima) of a function, are the largest and smallest value that the function takes at a point within a given interval.
- It may not be the minimum or maximum for the whole function, but **locally** it is.
- The term **local extremum** is used to denote either a local maximum or a local minimum.



- To define a local maximum, we need to consider an interval.
- Then a **local maximum** is the point where, the height of the function at **a** is greater than (or equal to) the height anywhere else in that interval.
- Or, more briefly:

$$\mathbf{f(a) \geq f(x) \text{ for all } x \text{ in the interval}}.$$

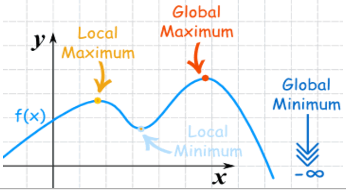


- To define a local minimum, we need to consider an interval.
- Then a **local minimum** is the point where, the height of the function at **a** is lowest than (or equal to) the height anywhere else in that interval.
- Or more briefly:

$$\mathbf{f(a) \leq f(x) \text{ for all } x \text{ in the interval}}.$$

Global maximum and global minimum

- The maximum or minimum over the entire function is called an **absolute** or **global** maximum or minimum.
- There is **only one** global maximum.
- And also there is **only one** global minimum.
- But there can be **more than one** local maximum or minimum.



Theorem 5.2
Local Extremum Theorem

Let f be defined on an interval $[a, b]$. If f has a local extremum at c, where $a < c < b$, and if f is differentiable at c, then $f'(c) = 0$.

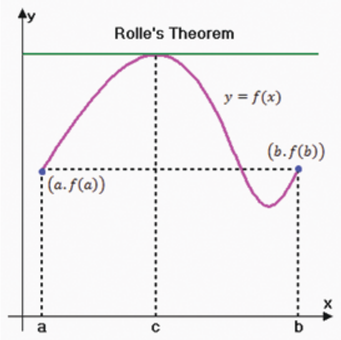
N.B: The Local Extremum Theorem does not make any assertion about a local extremum that occurs at a point c that is one of the end-points of the interval $[a, b]$.

Corollary 2

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then the extremum of f on $[a, b]$ can occur only at a, at b, or at points c in (a, b) where $f'(c) = 0$.

Theorem 5.3
Rolle's Theorem

Let f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there exists some point c, with $a < c < b$, for which $f'(c) = 0$.



If f is constant on $[a, b]$, then $f'(x) = 0$ everywhere in (a, b); in this case, we may take c to be any point of (a, b).

If f is non-constant on $[a, b]$, then either the maximum or the minimum (or both) of f on $[a, b]$ is different from the common value $f(a) = f(b)$.

Since one of the extrema occurs at some point c with $a < c < b$, the Local Extremum Theorem applied to the point c shows that $f'(c)$ must be zero.

Example 1

Verify that the conditions of Rolle's Theorem are satisfied by the function:

$$f(x) = x^4 - 2x^2, \quad x \in [-2, 2],$$

and determine a value of c in (-2,2) for which $f'(c) = 0$.

Example 1
Solution

Since f is a polynomial function, f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Also, $f(-2) = 8 = f(2)$. Thus, f satisfies the conditions of Rolle's Theorem on $[-2, 2]$.

It follows that there exists a number $c \in (-2, 2)$ for which $f'(c) = 0$. Now

$$\begin{aligned} f(x) &= x^4 - 2x^2, \\ f'(x) &= 4x^3 - 4x = 4x(x^2 - 1). \end{aligned}$$

So that f' vanishes at the points $x = -1$, $x = 0$, and $x = 1$ in (-2,2). Any of these three numbers will serve for c.

Example 1
Solution ⇒ Cont...

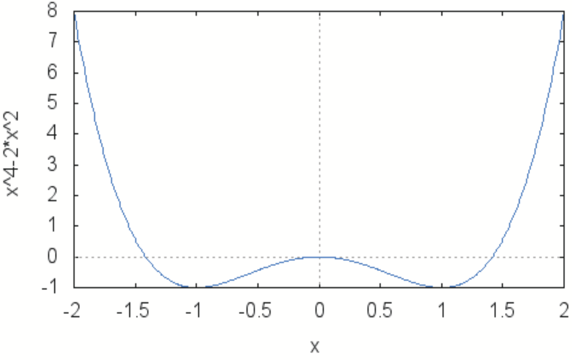


Figure: The graph of $f(x) = x^4 - 2x^2$

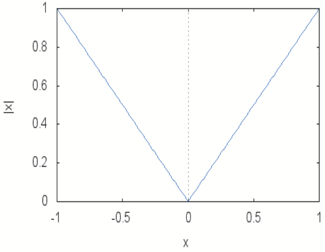
Example 2

Check that the conditions of Rolle's Theorem are satisfied by the function or not on [-1,1]:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Example 2
Solution

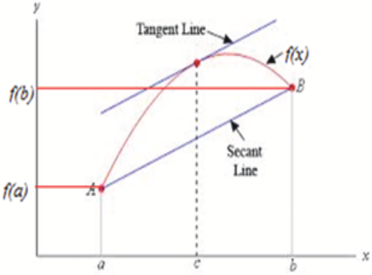
We have $f(-1) = 1 = f(1)$ and function is continuous on [-1,1]. However, it is not differentiable at $x = 0$. Hence, it is not differentiable on $(-1, 1)$. So the Rolle's Theorem cannot be applied.



Mean Value Theorem (MVT)

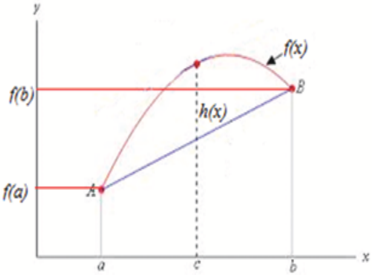
Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Mean Value Theorem (MVT)
The idea of the proof

We define $h(x)$ to be the verticle distance from the chord to the curve; then $h(a)$ and $h(b)$ are both 0; in fact, h satisfies all the conditions of Rolle's Theorem. Applying Rolle's Theorem to h, we obtain the desired result.



The slope of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is

$$m = \frac{f(b) - f(a)}{b - a},$$

and so the equation of the chord is

$$y = m(x - a) + f(a)$$

It follows that the verticle height, $h(x)$, between points with ordinates x on the graph and those on the chord is given by

$$h(x) = f(x) - [m(x - a) + f(a)].$$

Now $h(a) = h(b) = 0$, and h is continuous on $[a, b]$ and differentiable on (a, b). Thus h satisfies all the condtions of Rolle's Theorem.

It follows from the Rolle's Theorem that there exists some point c in (a, b) for which $h'(c) = 0$.

But, since $h'(c) = f'(c) - m$, it follows that

$$f'(c) = m = \frac{f(b) - f(a)}{b - a}.$$

Mean Value Theorem (MVT)
Remark

The special case, when $f(a) = f(b)$ is known as Rolle's Theorem. In this case, we have $f'(c) = 0$.

Example

Verify that the conditions of the Mean Value Theorem are satisfied by the function $f(x) = x^3 - x^2 - 2x$ on $[-1, 1]$; and find a value for c that satisfies the conclusion of the theorem.

Example
Solution

The function f is a polynomial function, f is continuous on $[-1, 1]$ and differentiable $(-1, 1)$. Thus, f satisfies the conditions of MVT. Now

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ &= \frac{-2 - 0}{1 - (-1)} \\ &= -1 \end{aligned} \tag{3}$$

$$\begin{aligned} f'(x) &= 3x^2 - 2x - 2 \\ f'(c) &= 3c^2 - 2c - 2. \end{aligned} \tag{4}$$

From (3) and (4) we have

$$3c^2 - 2c - 2 = -1 \Rightarrow c = -\frac{1}{3}, c = 1.$$

L'Hospital's Rule

- In Analysis and in Mathematical Physics we often need to evaluate limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}, \quad \text{where } f(a) = g(a) = 0.$$

- Such limits cannot be evaluated by the Qyotient Rule for limits of functions, because it does not apply in this situation.

- For example, do the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos 3x}{\sin x - e^{\cos x}}$$

exists? If they do, what are thier values?

Theorem
L'Hospital's Rule

Let f and g be differentiable on a neighbourhood of the point a, at which $f(a) = g(a) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists and equals } \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that this last limit exists.

Example

Find the following limits:

(a) $\lim_{x\to 0} \dfrac{\sin x}{2x}$

(b) $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 5x}$

(c) $\lim_{x\to \frac{\pi}{2}} \dfrac{\cos 3x}{\sin x - e^{\cos x}}$

Example
Solution

(a) Let $f(x) = \sin x$ and $g(x) = 2x$. Then f and g are differentiable on $\mathbb{R}$ and $f(0) = g(0) = 0$; so that f and g satisfy the conditions of L'Hospital's Rule at $x = 0$. Now

$$\frac{f'(x)}{g'(x)} = \frac{\cos x}{2}.$$

Then, by L'Hospital's Rule, we have

$$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

(b) Let $f(x) = \sin 2x$ and $g(x) = \sin 5x$. Then f and g are differentiable on $\mathbb{R}$ and $f(0) = g(0) = 0$; so that f and g satisfy the conditions of L'Hospital's Rule at $x = 0$. Now

$$\frac{f'(x)}{g'(x)} = \frac{2\cos 2x}{5\cos 5x}.$$

Then, by L'Hospital's Rule, we have

$$\lim_{x\to 0} \frac{\sin 2x}{\sin 5x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{2\cos 2x}{5\cos 5x} = \frac{2}{5}.$$

Example

Solution ⇒ Cont...

(c) Let $f(x) = \cos 3x$ and $g(x) = \sin x - e^{\cos x}$. Then f and g are differentiable on $\mathbb{R}$ and $f(\frac{\pi}{2}) = g(\frac{\pi}{2}) = 0$; so that f and g satisfy the conditions of L'Hospital's Rule at $x = \frac{\pi}{2}$. Now

$$\frac{f'(x)}{g'(x)} = \frac{-3\sin 3x}{\cos x + \sin x \times e^{\cos x}}.$$

Then, by L'Hospital's Rule, we have

$$\begin{aligned}\lim_{x \to \frac{\pi}{2}} \frac{f(x)}{g(x)} &= \lim_{x \to \frac{\pi}{2}} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \to \frac{\pi}{2}} \frac{-3\sin 3x}{\cos x + \sin x \times e^{\cos x}} \\ &= \frac{3}{1} = 3.\end{aligned}$$

Thank you !