Calculus (Real Analysis I)

(MAT122β)

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Limit and Continuity of Functions

Functions

- A function relates **each element** of a set X with **exactly one** element of another set Y.
- The set X is called the **domain** of the function.
- The range of the function is the set of element in Y that are assigned by this rule.

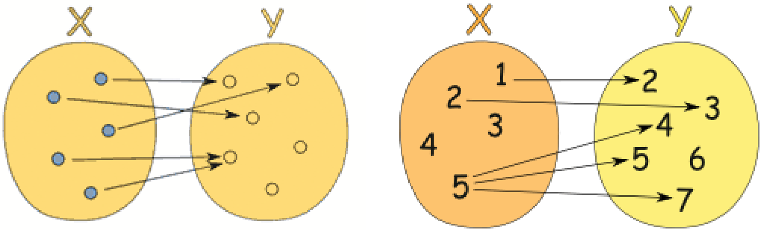
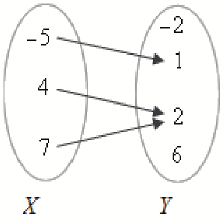


Figure: Left one is a function but right one is not a function.

Illustrative example

The following mapping diagram is an example of a function.



Each element of the set $X = \{-5, 4, 7\}$ is assigned to exactly one element of the set $Y = \{-2, 1, 2, 6\}$.

The domain of the function is $\{-5, 4, 7\}$ and the range of the function is $\{1, 2\}$.

Defining a function by an equation

The equation $y = 5x + 2$ defines y as a function of x since for each real number x, the expression $5x + 2$ is a unique real number.

However, not every equation in the variables x and y defines a function.

The equation $y^2 = x - 2$ does not define y as a function of x. Note that if $x = 11$, then $y^2 = 9$. This means that y could be either 3 or -3. Thus, more than one y value is assigned to $x = 11$.

The function notation

If we know that y is a function of x, we can use the function notation: $y = f(x)$.

Eg:

$$y = 5x^2 + 2x + 3 \Rightarrow f(x) = 5x^2 + 2x + 3.$$

Example

Decide whether or not each equation defines y as a function of x:

(a) $3x^2 + y^2 = 7$

(b) $y = 5x^2 + 4x + 9$

Example
Solution

(a) $3x^2 + y^2 = 7$ does not define y as a function of x.

(b) $y = 5x^2 + 4x + 9$ defines y as a function of x.

A **one-to-one** (**injective**) function f from set X to set Y is a function such that each x in X is related to a different y in Y.

More formally, we can restate this definition as either:

$f : X \to Y$ is one-to-one provided

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2,$$

or $f : X \to Y$ is one-to-one provided

$$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

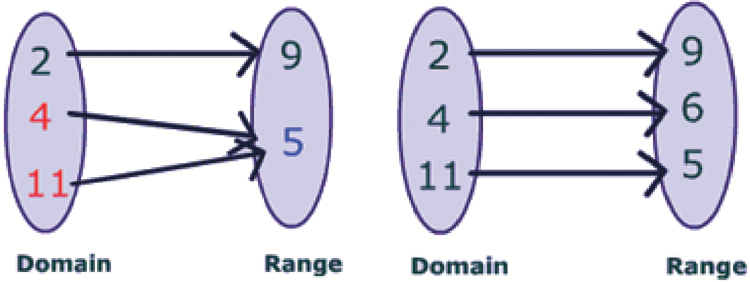


Figure: The left function is not one-to-one but right is one-to-one.

(a) Show that the function $f(x) = 3x + 2$ is one-to-one:

$$\begin{aligned}
\text{Assume } f(x_1) &= f(x_2) \\
3x_1 + 2 &= 3x_2 + 2 \\
3x_1 &= 3x_2 \\
x_1 &= x_2 \Leftarrow \text{ one-to-one}
\end{aligned}$$

(b) Show that the function $f(x) = x^2$ is not one-to-one:

$$\begin{aligned}
\text{Assume } f(x_1) &= f(x_2) \\
x_1^2 &= x_2^2 \\
x_1 &= \pm x_2 \Leftarrow \text{ not one-to-one}
\end{aligned}$$

Definition
Onto functions

A function $f : X \to Y$ is said to be **onto** (**surjective**) if for every y in Y, there is an x in X such that $f(x) = y$.

This can be restated as:

A function is onto when its image equals its range, i.e. $f(X) = Y$.

Onto functions
Illustrative examples

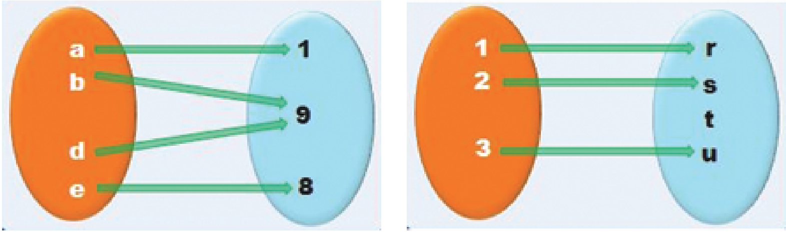


Figure: The left function is onto but right is not onto.

1. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 5$ is onto.

   Let y be in $\mathbb{R}$. Then $(y+5)$ and $(y+5)/3$ are also real numbers. Now

   $$f((y+5)/3) = 5[(y+5)/3] - 5 = y.$$

   Hence if y is in $\mathbb{R}$, there exists an x in $\mathbb{R}$ such that $f(x) = y$.

2. Is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/x$ onto?

   No! There is no x in $\mathbb{R}$ that has output 0.

Piecewise-defined function

- A piecewise-defined function applies different rules, usually as formulas, to disjoint (non-overlapping) subsets of its domain (subdomains).
- To evaluate such a function at a particular input value, we need to figure out which rule applies there.
- To graph such a function, we need to know how to graph the pieces that correspond to the different rules on their subdomains.

The piecewise definition of $f(x) = |x|$ is given by:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

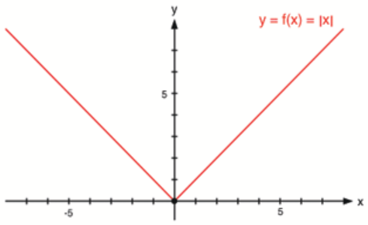


Figure: The graph of $y = |x|$.

Graph the following piecewise-defined function:

$$f(x) = \begin{cases} x+3 & \text{if } x \neq 3 \\ 7 & \text{if } x = 3. \end{cases}$$

- We graph the line $y = x + 3$, except we leave a hole at the point (3,6), since 3 is deleted from the subdomain of the top rule.
- The bottom rule defines $f(3)$ to be 7, so we plot the point (3,7).

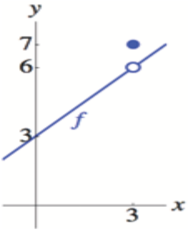


Figure: The graph of $f(x)$.

Limits of Functions

Why do we need limit?

- In some situations, we cannot work something out directly.
- But we can see how it behaves as we get closer and closer.
- Let's consider the function $f(x) = \frac{(x^2-1)}{(x-1)}$.
- Let's work it out for $x = 1$:

$$f(1) = \frac{(1^2-1)}{(1-1)} = \frac{0}{0}.$$

Why do we need limit?
Cont...

- We don't really know the value of $0/0$.
- So we need another way of answering this.
- The limits can be used to give an answer in such a situations.

Why do we need limit?
Cont...

Instead of trying to work it out for $x = 1$, let's try approaching it closer and closer from $x < 1$:

x	$\dfrac{(x^2-1)}{(x-1)}$
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
...	...

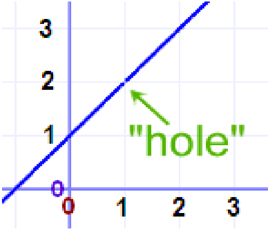
Why do we need limit?
Cont...

Instead of trying to work it out for x=1, let's try approaching it closer and closer from $x > 1$:

x	$\dfrac{(x^2-1)}{(x-1)}$
1.5	2.50000
1.1	2.10000
1.01	2.01000
1.001	2.00100
1.0001	2.00010
1.00001	2.00001
...	...

Why do we need limit?
Cont...

- Now we can see that as x gets close to 1, then $(x^2-1)/(x-1)$ gets close to 2.
- When $x=1$ we don't know the answer.
- But we can see that it is going to be 2.



Why do we need limit?
Cont...

- We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit".

  The limit of $(x^2 - 1)/(x - 1)$ as x approaches 1 is 2.

- It can be written symbolically as:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Combination Rules for limits

If $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$, then:

Sum Rule $\lim_{x \to c}(f(x) + g(x)) = l + m$

Multiple Rule $\lim_{x \to c}(\lambda f(x)) = \lambda l$, for $\lambda \in \mathbb{R}$

Product Rule $\lim_{x \to c}(f(x)g(x)) = lm$

Quotient Rule $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided that $m \neq 0$

Example

Find following limits:

(a) $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

(b) $\lim_{x \to -2} \dfrac{x^2 + 6x + 8}{x + 2}$

(c) $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x^2 - 9}$

Example
Solution

(a) $\lim_{x\to 3} \dfrac{x^2-9}{x-3} = 6$

(b) $\lim_{x\to -2} \dfrac{x^2+6x+8}{x+2} = 2$

(c) $\lim_{x\to 3} \dfrac{x^2-6x+9}{x^2-9} = 0$

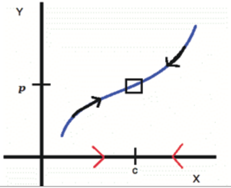
Right and left hand limits

If $f(x)$ approaches the value p as x approaches to c, we say p is the limit of the function $f(x)$ as x tends to c. That is

$$\lim_{x \to c} f(x) = p.$$

Then we can define right and left hand limit as follows:

$$\lim_{x \to c^-} f(x) = p \Leftarrow \text{Left Hand Limit},$$
$$\lim_{x \to c^+} f(x) = p \Leftarrow \text{Right Hand Limit}.$$



Example 1

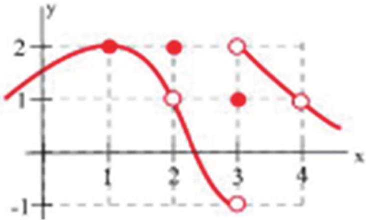
Use the graph to determine following limits:

(a) $\lim_{x\to 1} f(x)$

(b) $\lim_{x\to 2} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $\lim_{x\to 4} f(x)$



Example 1
Solution

(a) $\lim_{x\to 1} f(x) = 2$

(b) $\lim_{x\to 2} f(x) = 1$

(c) $\lim_{x\to 3} f(x) \Leftarrow$ does not exist

(d) $\lim_{x\to 4} f(x) = 1$

Example 2

The function f is defined by:

$$f(x) = \begin{cases} x+3 & \text{if } x \leq 2 \\ -x+7 & \text{if } x > 2. \end{cases}$$

What is $\lim_{x\to 2} f(x)$.

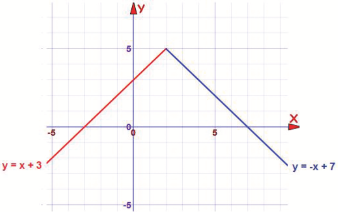
Example 2
Solution

Let's consider the left and right hand side limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+3 = 5$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x+7 = 5$$

We get same value for left and right hand limits. Hence

$$\lim_{x\to 2} f(x) = 5.$$

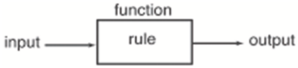


About definition of the limit

- The goal of this section is to provide a precise definition of the limit of a function.
- The definition cannot be used to calculate the values of limits, but it provides a statement of what a limit is.
- The definition of limit is then used to verify the limits of some functions, and some general results are proved.
- But before stating the definition of limit of a function, we shall consider similarities between sequences and functions.

What sequence and function have in common?

- A function is a rule which operates on an input and produces an output.
- A sequence is a type of function.
- A sequence is a function with integers at or greater than zero as input.



What is limit of a sequence?

- Consider the sequence where $x_n = \frac{1}{n},\ n = 1, 2, \cdots.$
- The numbers in this sequence get closer and closer to zero.
- Whatever positive number we choose, x_n will eventually become smaller than that number, and stay smaller.
- So x_n eventually gets closer to zero than any distance we choose, and stays closer.
- We say that the sequence x_n has limit zero as n tends to infinity.

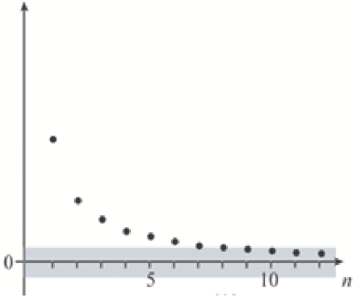


Figure: The graph of the sequence $\frac{1}{n}$.

What is limit of a function?

- We define the limit of a function in a similar way.
- For example, the points of the sequence $\{\frac{1}{n}\}$ are also points on the graph of the function $\frac{1}{x}$ for $x > 0$.
- As x gets larger, $f(x)$ gets closer and closer to zero.
- In fact, $f(x)$ will get closer to zero than any distance we choose, and will stay closer.
- We say that $f(x)$ has limit zero as x tends to infinity, and we write $\lim_{x\to\infty} f(x) = 0$.

What is limit of a function?
Cont...

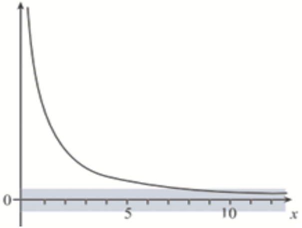


Figure: The graph of the function $\frac{1}{x}$.

Motivative example for definition

- Consider the limit: $\lim_{x \to 3} 2x - 1 = 5$.
- As the values of x approaches 3, the values of $2x - 1$ approaches 5.
- When x is close to 3 (but not equal to 3), the value of $2x - 1$ is close to 5.
- We can guarantee that the value of $f(x) = 2x - 1$ are close to 5 as we want by starting with values of x sufficiently close to 3 (but not equal to 3).

Motivative example for definition
Cont...

We know that $\lim_{x\to 3} 2x - 1 = 5$. So that we can guarantee that the values of $f(x) = 2x - 1$ are as close to 5 as we want by starting with values of x sufficiently close to 3. What values of x guarantee that $f(x) = 2x - 1$ is within

(a) 1 unit of 5?

(b) 0.2 units of 5?

(c) E units of 5?

Motivative example for definition

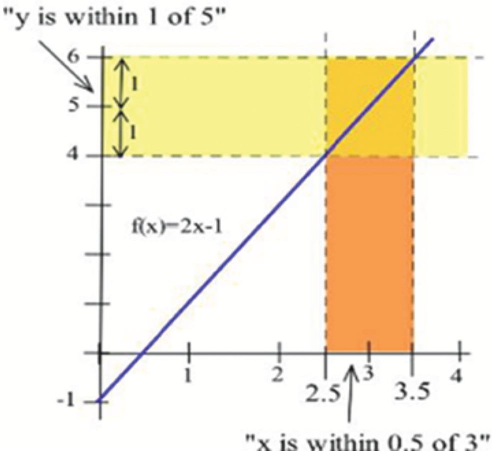
Cont...

(a)

- Within 1 unit of 5 means between 5-1=4 and 5+1=6, so the question can be rephrased as "for what values of x is $y = 2x - 1$ between 4 and 6: $4 < 2x - 1 < 6$?".
- Straightforward calculation implies that $2.5 < x < 3.5$.
- We can restate this result as follows: "If x is within 0.5 units of 3, then $y = 2x - 1$ is within 1 unit of 5.
- Any smaller distance also satisfies the guarantee: "If x is within 0.4 units of 3, then $y = 2x - 1$ is within 1 unit of 5.

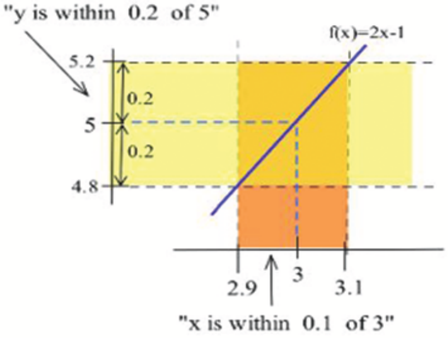
Motivative example for definition

Solution ⇒ Cont...



Motivative example for definition
Cont...

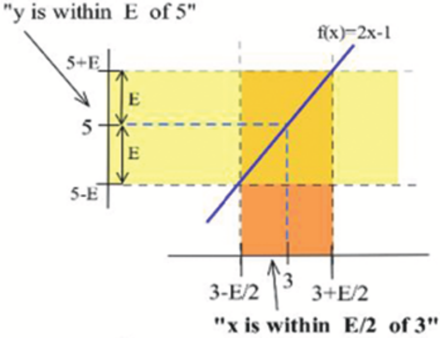
(b) "If x is within 0.1 units of 3, then $y = 2x - 1$ is within 0.2 unit of 5. Any smaller distance also satisfies the guarantee.



Motivative example for definition

Cont...

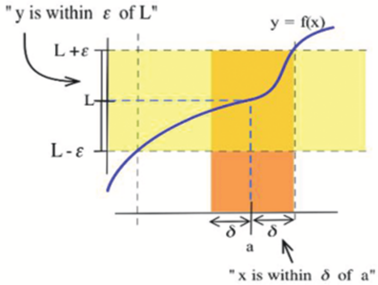
(c) "If x is within E/2 units of 3, then $y = 2x - 1$ is within E unit of 5. Any smaller distance also satisfies the guarantee.



Definition
Limit of a function

$\lim_{x\to a} f(x) = L$ means for every given $\epsilon > 0$ there is a $\delta > 0$ so that if x is within δ units of a (and $x \neq a$) then $f(x)$ is within ϵ units of L.

Equivalently $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.



Example 1

Using the $\epsilon - \delta$ definition of the limit of a function show that:

$$\lim_{x \to 3} 4x - 5 = 7.$$

We need to show that "for every given $\epsilon > 0$ there is a $\delta > 0$ so that if x is within δ units of 3 (and $x \neq 3$) then $4x - 5$ is within ϵ units of 7".

Actually there are two things we need to do.

1. First, we need to find a value for δ (typically depending on ϵ).
2. Second, we need to show that our δ really does satisfy the "if then" part of the definition.

To find value for δ, assume that $4x - 5$ is within ϵ units of 7 and solve for x. If

$$\begin{aligned} 7 - \epsilon &< 4x - 5 < 7 + \epsilon \\ 12 - \epsilon &< 4x < 12 + \epsilon \\ 3 - \frac{\epsilon}{4} &< x < 3 + \frac{\epsilon}{4} \end{aligned}$$

So x is within $\frac{\epsilon}{4}$ units of 3. Put $\delta = \frac{\epsilon}{4}$.

To show that $\delta = \frac{\epsilon}{4}$ satisfies the definition, we merely reverse the order of the above steps.

Assume that x is within δ units of 3. Then

$$\begin{aligned} 3-\delta &< x < 3+\delta \\ 3-\frac{\epsilon}{4} &< x < 3+\frac{\epsilon}{4} \\ 12-\epsilon &< 4x < 12+\epsilon \\ 7-\epsilon &< 4x-5 < 7+\epsilon \end{aligned}$$

We can conclude that $f(x) = 4x - 5$ is within ϵ units of 7. This formally verifies that $\lim_{x\to 3} 4x - 5 = 7$.

Example 2

Using the $\epsilon - \delta$ definition of the limit of a function show that:

$$\lim_{x \to 4} 2x = 8.$$

Example 2
Solution

We need to show that "for every given $\epsilon > 0$ there is a $\delta > 0$ so that if x is within δ units of 4 (and $x \neq 4$) then $2x$ is within ϵ units of 8".

To find value for δ, assume that $2x$ is within ϵ units of 8 and solve for x. If

$$\begin{aligned} 8-\epsilon &< 2x < 8+\epsilon \\ 4-\frac{\epsilon}{2} &< x < 4+\frac{\epsilon}{2} \end{aligned}$$

So x is within $\frac{\epsilon}{2}$ units of 4. Put $\delta = \frac{\epsilon}{2}$.

To show that $\delta = \frac{\epsilon}{2}$ satisfies the definition, we merely reverse the order of the above steps.

Assume that x is within δ units of 3. Then

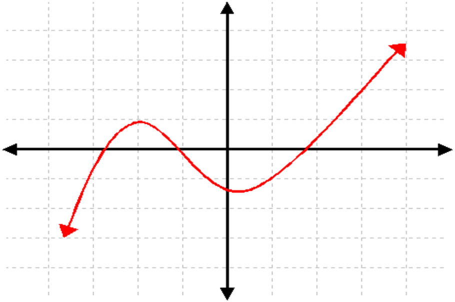
$$\begin{aligned} 4-\delta &< x < 4+\delta \\ 4-\frac{\epsilon}{2} &< x < 4+\frac{\epsilon}{2} \\ 8-\epsilon &< 2x < 8+\epsilon \end{aligned}$$

We can conclude that $f(x) = 2x$ is within ϵ units of 8. This formally verifies that $\lim_{x\to 4} 2x = 8$.

Continuity of Functions

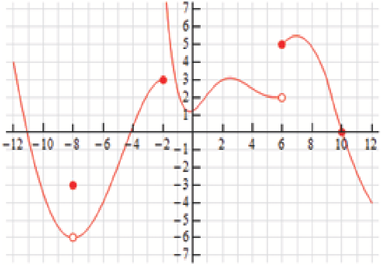
What is a continuous function?

- A function is continuous when its graph is a single unbroken curve.
- Simply, a function is continuous at a point if it does not have a break or gap in its value at that point.



What is a discontinuous function?

- All functions are not continuous.
- If a function is not continuous at a point in its domain, one says that it has a discontinuity there.



A function $f(x)$ defined in a neighborhood of a point c and also at c is said to be continuous at $x = c$, if

$$\lim_{x \to c} f(x) = f(c).$$

From the above definition it follows that the following three conditions are necessary for the function $f(x)$ to be continuous at point c:

1. $f(x)$ is defined at $x = c$, that is $f(c)$ exists,
2. $\lim_{x \to c} f(x)$ exists,
3. $\lim_{x \to c} f(x) = f(c)$.

Strategy for continuity

Any function is said to be continuous at $x = c$ iff right side limit is equal to left side limit and both are equal to the value of the function at that point.

Example 1

Show that the following function is not continuous at $x = 2$:

$$f(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ x^2 + x & \text{if } x > 2. \end{cases}$$

Example 1

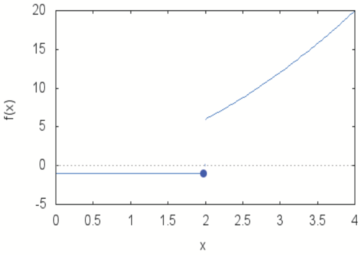
Solution

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} -1 = -1 \tag{1}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + x = 6 \tag{2}$$

Since $(1) \neq (2)$, the limit does not exist even though $f(2) = -1$.

Hence f is not continuous at $x = 2$.



Example 2

Check the continuity of the following function at $x = 0$:

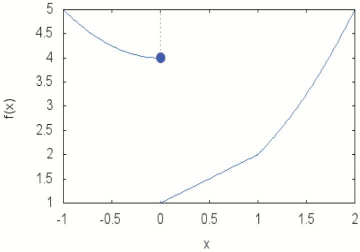
$$f(x) = \begin{cases} x^2 + 4 & \text{if } x \leq 0 \\ x + 1 & \text{if } 0 < x \leq 1 \\ x^2 + 1 & \text{if } x > 1. \end{cases}$$

Example 2
Solution

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2+4=4 \tag{3}$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1=1 \tag{4}$$

Since (3) $\neq$ (4) the limit does not exist even though $f(0)=4$.

Hence f is not continuous at $x=0$.



Example 3

Check the continuity of the function:

$$f(x) = \begin{cases} \dfrac{x^3 - 1}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Example 3
Solution

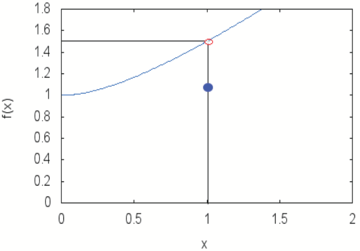
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3-1}{x^2-1} = \frac{3}{2} \tag{5}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3-1}{x^2-1} = \frac{3}{2} \tag{6}$$

Since (5) = (6), the limit exist at $x = 1$.

But $\lim_{x\to 1} f(x) \neq f(1)$.

Hence f is not continuous at $x = 1$.



Exercise

Let f be given by

$$f(x) = \begin{cases} x^2 - x + 1 & \text{if } x \leq 1 \\ kx^2 + 1 & \text{if } x > 1. \end{cases}$$

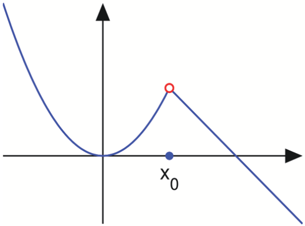
For which value of k is f continuous on its domain?

Classification of discontinuities
Removable discontinuity

Consider the function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 2 - x & \text{if } x > 1. \end{cases}$$

Then, the point $x_0 = 1$ is a removable discontinuity.

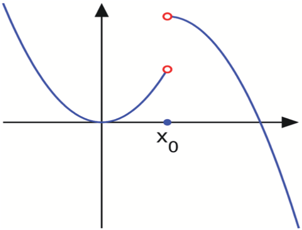


Classification of discontinuities
jump discontinuity

Consider the function:

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ 2 - (x-1)^2 & \text{if } x > 1. \end{cases}$$

Then, the point $x_0 = 1$ is a jump discontinuity.



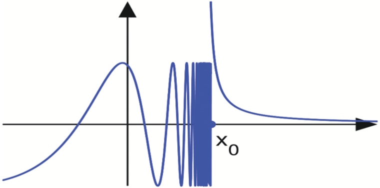
Classification of discontinuities

Essential discontinuity

Consider the function:

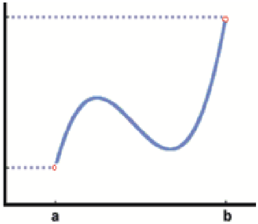
$$f(x) = \begin{cases} \sin\left(\dfrac{5}{x-1}\right) & \text{if } x < 1 \\ 0 & \text{if } x = 1 \\ \dfrac{1}{x-1} & \text{if } x > 1. \end{cases}$$

Then, the point $x_0 = 1$ is an essential discontinuity (sometimes called infinite discontinuity). For it to be an essential discontinuity, it would have sufficed that only one of the two one-sided limits did not exist or were infinite.



Continuity on an open interval

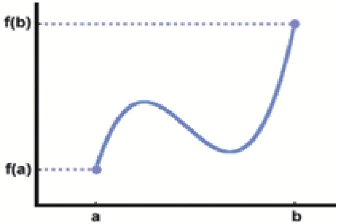
- A function f is continuous on the open interval (a, b) if f is continuous at every point in (a, b).
- This can also be extended to unbounded open intervals of the form (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$ as well.



Continuity on a closed interval

A function f is continuous on the closed interval $[a, b]$ if and only if:

1. f is defined on $[a, b]$,
2. f is continuous on (a, b),
3. $\lim_{x \to a^+} f(x) = f(a)$, and
4. $\lim_{x \to b^-} f(x) = f(b)$.



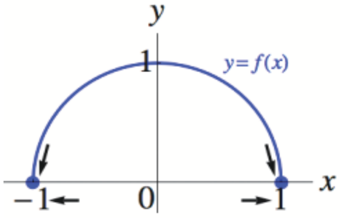
Example

Let $f(x) = \sqrt{1 - x^2}$. Show that f is continuous on the closed interval $[-1, 1]$.

Example
Solution

$$\begin{aligned} y &= \sqrt{1-x^2} \\ y^2 &= 1-x^2 \qquad y \geq 0 \\ x^2+y^2 &= 1 \qquad y \geq 0 \end{aligned}$$

The graph is the upper half of the unit circle centered at the origin, including the points (-1, 0) and (1, 0).



Example
Solution ⇒ Cont...

The function f is continuous on $[-1, 1]$ because

1. f is defined on $[-1, 1]$,
2. f is continuous on $(-1, 1)$,
3. $\lim_{x\to -1^+} f(x) = f(-1)$, so f is continuous from the right at -1, and,
4. $\lim_{x\to 1^-} f(x) = f(1)$, so f is continuous from the left at 1.

Continuity of functions and convergence of sequences

- There are several ways of checking the continuity.
- Now we adopt a definition which involves the convergence of sequences, as this will enable us to use the result about sequences that we met in Chapter 3.

Definition

A function f defined on an interval I that contains c as an interior point is **continuous** at c if:

for each sequence $\{x_n\}$ in I such that $x_n \to c$, then $f(x_n) \to f(c)$.

If f is not continuous at the point c in I, then it is **discontinuous** at c.

Example 1

Prove that the function $f(x) = x^3,\ x \in \mathbb{R}$, is continuous at the point $\frac{1}{2}$.

Example 1
Solution

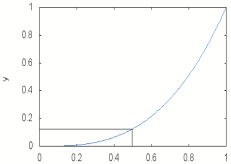
Let $\{x_n\}$ be any sequence in $\mathbb{R}$ that convergence to $\frac{1}{2}$, that is, $x_n \to \frac{1}{2}$. Then, by Combination Rules for sequences, it follows that

$$f(x_n) = x_n^3 \to \left(\frac{1}{2}\right)^3 = \frac{1}{8} \text{ as } n \to \infty,$$

while $f\left(\frac{1}{2}\right) = \frac{1}{8}$.

In otherwards, $\{f(x_n)\}$ converges to $f\left(\frac{1}{2}\right)$ as $n \to \infty$.

It follows that f is continuous at $\frac{1}{2}$, as required.



Example 2

Prove that the function:

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0, \end{cases}$$

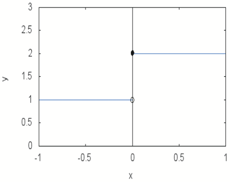
is discontinuous at 0.

Example 2
Solution

Chose the sequence $\{x_n\} = \{-\frac{1}{n}\},\ n = 1, 2, \cdots$. For this sequence

$$f(x_n) = 1 \to 1 \neq f(0) \text{ as } n \to \infty.$$

It follows that the function f cannot be continuous at 0.



Combination Rules for continuous functions

If f and g are functions that are continuous at point c, then so are:

Sum Rule	$f + g$
Multiple Rule	λf for $\lambda \in \mathbb{R}$
Product Rule	fg
Quotient Rule	$\dfrac{f}{g}$, provided that $g(c) \neq 0$

Thank you !