Calculus (Real Analysis I)

(MAT122β)

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Sequences

Why do we need sequences?

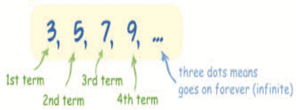
- Sequences are useful in studying functions, spaces, and other mathematical structures using the convergence properties of sequences.
- In particular, sequences are the basis for series, which are important in differential equations and analysis.
- Sequences are also of interest in their own right and can be studied as patterns or puzzles, such as in the study of prime numbers.

What is a sequence?

- A Sequence is a set of objects (usually numbers) that are in order.
- The objects are "in order" means we are free to define what order that is.
- They could go forwards, backwards or they could alternate.

Finite and infinite sequensces

- A sequence contains members and they are also called elements, or terms.
- If a sequence contains an infinite number of members it is called an **infinite sequence**, otherwise it is a **finite sequence**.



1, 3, 5, 7, 9, 11, 13, 15

(i) $\{1, 2, 3, 4, \cdots\} \Leftarrow$ An infinite sequence

(ii) $\{2, 4, 6, 8\} \Leftarrow$ A finite sequence

(iii) $\{6, 9, 12, 15, 18, \cdots\} \Leftarrow$ An infinite sequence

(iv) $\{4, 3, 2, 1\} \Leftarrow$ A finite sequence

(v) $\{a, b, c, d, e\} \Leftarrow$ A finite sequence

(vi) $\{0, 1, 0, 1, 0, 1, \cdots\} \Leftarrow$ An alternating infinite sequence

Difference between a set and a sequence

- The terms of a sequence are in order. But the order of terms is not a concern for a set.
- The same value can appear many times in a sequence. But in a set, a value can appear only once.

Difference between a set and a sequence
Example

- $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$ is a sequence. The corresponding set would be $\{0, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$.
- $\{0, 1, 0, 1, 0, 1, \cdots\}$ is the sequence of alternating 0s and 1s. The corresponding set would be just $\{0, 1\}$.

Indexing notation

- The terms of a sequence are usually denoted like x_n, with the subscripted letter n being the "index".
- So the second term of a sequnce might be named x_2 and x_{10} would designate the tenth term.
- Sometimes sequences start with an index of $n = 0$, so the first term would be x_0 and the second term would be x_1.

- The indexing notation is a natural notation for sequences whose elements are related to the index n in a simple way.
- Sequences can be indexed beginning and ending from any integer.
- For instance, a finite sequence can be denoted by $\{x_n\}_{n=1}^{k}$.
- The infinity symbol ∞ is often used as the superscript to indicate the sequence including all integer.
- So, an infinite sequence can be denoted by $\{x_n\}_{n=1}^{\infty}$ or (x_n) or $(x_n)_{n\in\mathbb{N}}$.

Indexing notation
Cont...

(i) The sequence of the first 10 square numbers could be written as $\{x_n\}_{n=1}^{10}, \quad x_n = n^2$.

(ii) The sequence of all positive squares is then denoted $\{x_n\}_{n=1}^{\infty}, \quad x_n = n^2$.

Rule of a sequence

- A sequence usually has a rule, which is a way to find the value of each term.
- So, the rule should be a formula with n in it (where n is any term number).

Find missing numbers in following sequences:

(i) $3, 5, 7, 9, 11, 13, \_$

(ii) $1, 4, 9, 16, \_$

(iii) $3, 5, 8, 13, 21, \_$

(iv) $1, 4, 7, 10, 13, 16, 19, 22, 25, \_$

Rule of a sequence
Cont...

Sequence	**Rule**
$3, 5, 7, 9, 11, 13, \cdots$	$x_n = 2n + 1$
$1, 4, 9, 16, \cdots$	$x_n = n^2$
$3, 5, 8, 13, 21, \cdots$	$x_n = x_{n-1} + x_{n-2}$
$1, 4, 7, 10, 13, 16, 19, 22, 25, \cdots$	$x_n = 3n - 2$

Examples

Write down first five terms of the following sequences.

(i) $\{2n+1\}$

(ii) $\{3^{-n}\}$

(iii) $\{(1+\frac{1}{n})^n\}$

(iv) $\{(-1)^n\}$

(v) $\{(-1/n)^n\}$

(i) $\{3, 5, 7, 9, 11\}$

(ii) $\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \frac{1}{3^5}\}$

(iii) $\{(1+\frac{1}{1})^1, (1+\frac{1}{2})^2, (1+\frac{1}{3})^3, (1+\frac{1}{4})^4, (1+\frac{1}{5})^5\}$

(iv) $\{-1, 1, -1, 1, -1\}$

(v) $\{-1, 1/4, -1/27, 1/256, -1/3125\}$

- In an arithmetic sequence the difference between one term and the next is a constant.
- If the first term is a and the common difference is d, then the corresponding arithmetic sequence can be written down as $\{a, a+d, a+2d, a+3d, \cdots\}$.
- The rule for the arithmetic sequence is $x_n = a + d(n-1)$.

- In a geometric sequence each term is found by multiplying the previous term by a constant.
- If the first term is a and the common ratio is r, then the corresponding geometric sequence can be written down as $\{a, ar, ar^2, ar^3, \cdots\}$.
- The rule for the geometric sequence is $x_n = ar^{(n-1)}$.

- In mathematics, the Fibonacci sequence are the numbers in the integer sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$.
- The next number is found by adding the two numbers before it together.
- The rule for the Fibonacci sequence is $x_n = x_{n-1} + x_{n-2}$.

Monotonic sequences

- Many sequences have the property that, as n increases thier terms are either **increasing** or **decreasing**.
- For example, the sequence $\{2n+1\}$ has terms $3, 5, 7, 9, \cdots$, which are increasing, whereas the sequence $\{\frac{1}{n}\}$ has terms $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ which are decreasing.
- The sequence $\{(-1)^n\}$ is neither increasing nor decreasing.
- We now give a precise meaning to these words **increasing** and **decreasing**, and introduce the term **monotonic**.

A sequence $\{x_n\}$ is

1. **constant**, if $x_{n+1} = x_n$, for $n = 1, 2, 3, \cdots$;
2. **increasing**, if $x_{n+1} \geq x_n$, for $n = 1, 2, 3, \cdots$;
3. **decreasing**, if $x_{n+1} \leq x_n$, for $n = 1, 2, 3, \cdots$;
4. **monotonic**, if $\{x_n\}$ is either increasing or decreasing.

A sequence $\{x_n\}$ is

1. **strictly increasing**, if $x_{n+1} > x_n$, for $n = 1, 2, 3, \cdots$;
2. **strictly decreasing**, if $x_{n+1} < x_n$, for $n = 1, 2, 3, \cdots$;
3. **strictly monotonic**, if $\{x_n\}$ is either strictly increasing or strictly decreasing.

Strictly monotonic sequences
Example 1

1. $2, 2, 2, 2, \cdots \Leftarrow$ constant sequence
2. $1, 2, 3, 4, \cdots \Leftarrow$ strictly increasing sequence
3. $4, 3, 2, 1 \cdots \Leftarrow$ strictly decreasing sequence
4. $1, 1, 2, 2, 3, 3, 4, 4, \cdots \Leftarrow$ increasing but not strictly

Strictly monotonic sequences
Example 2

Determine which of the following sequences $\{x_n\}$ are monotonic:

(a) $x_n = 2n - 1, n = 1, 2, \cdots$

(b) $x_n = \frac{1}{n}, n = 1, 2, \cdots$

(c) $x_n = (-1)^n, n = 1, 2, \cdots$

(a) The sequence $\{2n-1\}$ is monotonic because $x_n = 2n-1$ and $x_{n+1} = 2(n+1)-1 = 2n+1$, so that

$$x_{n+1} - x_n = (2n+1) - (2n-1) = 2 > 0, \text{ for } n = 1, 2, \cdots$$

Thus $\{2n-1\}$ is increasing.

In fact, strictly increasing.

(b) The sequence $\{\frac{1}{n}\}$ is monotonic because $x_n = \frac{1}{n}$ and $x_{n+1} = \frac{1}{n+1}$, so that

$$x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{(n+1)n} < 0, \text{ for } n = 1, 2, \cdots$$

Thus $\{\frac{1}{n}\}$ is decreasing.

(c) The sequence $\{(-1)^n\}$ is not monotonic.

In fact, $x_1 = -1$, $x_2 = 1$ and $x_3 = -1$.

Hence $x_3 < x_2$, which means that $\{x_n\}$ is not increasing.

Also, $x_2 > x_1$, which means that $\{x_n\}$ is not decreasing.

Thus $\{(-1)^n\}$ is neither increasing nor decreasing, and so is not monotonic.

Bounded and unbounded sequences

- A sequence $x_1, x_2, x_3, \cdots$ is bounded if there exist a number $M > 0$ such that

$$|x_n| \leq M$$

for every natural number n. Otherwise we call sequence is unbounded.

- This means that regardless of what term we are looking at, the absolute value of that term must be less than M.

Bounded and unbounded sequences

Example 1

Consider the sequence $\{2^{-n}\}_{n=1}^{\infty}$.

The first few terms of the sequence are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots$.

It is clear that any value greater than $\frac{1}{2}$ will bound this sequence.

Therefore, this is an example for a bounded sequence.

Consider the sequence

$$a_n = \begin{cases} 1 + \frac{1}{2^n} & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}$$

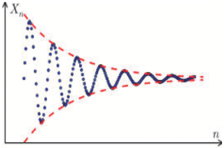
The first few terms of the sequence are $\frac{3}{2}, 4, \frac{9}{8}, 16, \cdots$.

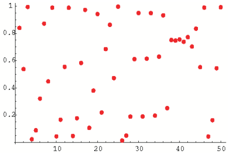
Regardless of the value of M that you choose it is still possible to find a value of n such that $2^n > M$.

Therefore this sequence is unbounded.

Convergent sequences and divergent sequences

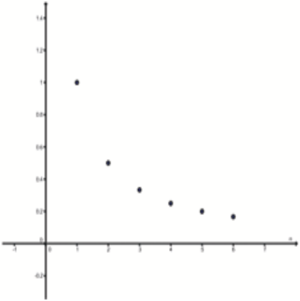
- The **limit** of a sequence is the value that the terms of a sequence "tend to".
- If such a limit exists, the sequence is called **convergent**.
- A sequence which does not converge is said to be **divergent**.





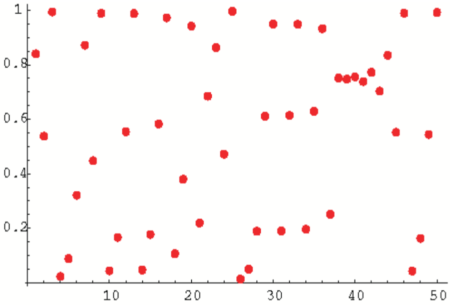
- Consider the sequence $x_n = \{\frac{1}{n}\}$.
- The first few terms of the sequence can be written down as $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots\}$.
- Examining the list above, the terms of the sequence seem to be getting smaller and smaller and appear to be tending towards 0.
- We might speculate that the sequence $\{x_n\}$ converges to 0.

- Visually we can see that the sequence seems to be converging towards 0.
- That is, as n gets bigger, $\{x_n\}$ gets closer to 0.



- Consider the sequence $x_{n+1} = \{4(1 - x_n)x_n\}$, where $x_1 = 0.3$.
- The first few terms of the sequence can be written down as $\{0.3, 0.84, 0.5376, 0.994345, 0.0224922,$ $0.0879454, 0.320844, 0.871612 \cdots\}$.
- Examining the list above, we can see that sequence does not tend to any value.
- We might speculate that the sequence $\{x_n\}$ diverges.

- The below Figure shows the first 50 terms of the above sequence.
- Even if we compute the first billion terms nothing nice will happen.



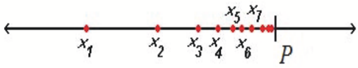
How do we identify convergent sequences?

- In general we need to examine the behaviour of the sequence as n gets bigger and bigger, in order to determine if it converges or diverges.
- What we have done above is not sufficient to prove a sequence x_n converges to a value.
- We will see that the way we examine the behaviour of a sequence x_n, as n gets bigger and bigger, is to take the $\lim_{n\to\infty} x_n$.

The idea of convergence

The idea of convergence of a sequence is made precise with the use the distance on $\mathbb{R}$.

Keep in mind that x_n comes near and near a point p means that the distance $|x_n - p|$ keeps steadily getting less and less as n increases.



A sequence $\{x_n\}$ is said to converge to a number p if for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|x_n - p| < \epsilon$ for every $n \geq n_0$.

In that case,the number p is said to be **limit** of the sequence $\{x_n\}$, and we write,

$$x_n \to p, \quad \text{or} \quad \lim_{n\to\infty} x_n = p \quad \text{or} \quad \lim x_n = p.$$

If $\{x_n\}$ converges, it is called a **convergent sequence** and if sequence $\{x_n\}$ does not converge, it is said to be a **divergent** sequence and we say $\{x_n\}$ diverges.

$x_n \to p$ if given $\epsilon > 0$, from some n_0^{th}-term onwards all the terms of the sequence are at a distance $< \epsilon$ from p.

In general, checking convergence amounts to finding suitable n_0 for any given $\epsilon > 0$, which will satisfy the requirement $|x_n - p| < \epsilon$, for all $n \geq n_0$.

For different value of ϵ, the cut-off point n_0 will in general, be different.

Example 1

Show that the constant sequence $\{c, c, c, \cdots\}$ converges to c.

Example 1
Solution

- For each n, $x_n = c$ and given any $\epsilon > 0$, for any choice of n_0, $|x_n - c| = 0 < \epsilon$.
- Therefore, the costant sequence $\{c, c, c, \cdots\}$ converges to c.

Example 2

Consider the sequence $\{\frac{1}{n}\}$ and show that $\lim_{n\to\infty} x_n = 0$.

Example 2
Solution

- Given an $\epsilon > 0$, let us choose a n_0 such that $\frac{1}{n_0} < \epsilon$.
- Now, if $n \geq n_0$, then we have

$$\begin{aligned} |x_n - 0| &= \left|\frac{1}{n} - 0\right| \\ &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \leq \frac{1}{n_0} < \epsilon. \end{aligned}$$

- This is exactly what we needed to show to conclude that $\lim_{n\to\infty} x_n = 0$.

Example 3

Consider the sequence $\{\frac{n+1}{n}\}$ and show that $\lim_{n\to\infty} x_n = 1$.

Example 3
Solution

- Given an $\epsilon > 0$, let us choose a n_0 such that $\frac{1}{n_0} < \epsilon$.
- Now, if $n \geq n_0$, then we have

$$\begin{aligned} |x_n - 1| &= \left|\frac{n+1}{n} - 1\right| \\ &= \left|\frac{1}{n}\right| \\ &= \frac{1}{n} \leq \frac{1}{n_0} < \epsilon. \end{aligned}$$

- This is exactly what we needed to show to conclude that $\lim_{n\to\infty} x_n = 1$.

Given $\epsilon > 0$, we notice that

$$\left|\frac{n+1}{n} - 1\right| = \frac{1}{n} < \epsilon \text{ whenever } n > \frac{1}{\epsilon}.$$

Thus, n_0 should be some natural number larger than $\frac{1}{\epsilon}$. For example, if $\epsilon = 1/99$, then we may choose n_0 to be any positive integer bigger than 99, and we conclude that

$$\left|\frac{n+1}{n} - 1\right| < \epsilon = \frac{1}{99} \text{ whenever } n \geq n_0 = 100.$$

Similarly, if $\epsilon = 2/999$, then $\frac{1}{\epsilon} = 499.5$, so that

$$\left|\frac{n+1}{n} - 1\right| < \epsilon = \frac{2}{999} \text{ whenever } n \geq n_0 = 500.$$

Thus, n_0 clearly depends on ϵ.

Example 4

Consider the sequence $x_n = \frac{(2n+1)}{(1-3n)}$ and show that $\lim_{n\to\infty} x_n = -\frac{2}{3}$.

Example 4
Solution

- Indeed, if $\epsilon > 0$ is given, we must find a n_0, such that if $n \geq n_0$ then

$$\left|x_n - (-\frac{2}{3})\right| = \left|x_n + \frac{2}{3}\right|.$$

- Let us examine the quantity $\left|x_n + \frac{2}{3}\right|$.

$$\begin{aligned}\left|x_n + \frac{2}{3}\right| &= \left|\frac{(2n+1)}{(1-3n)} + \frac{2}{3}\right| \\ &= \left|\frac{6n+3+2-6n}{3-9n}\right| \\ &= \left|\frac{5}{3-9n}\right|\end{aligned}$$

$$
\begin{aligned}
\left|x_n + \frac{2}{3}\right| &= \frac{5}{9n-3} \\
&= \frac{5}{6n+3n-3} \\
&\leq \frac{5}{6n} \\
&< \frac{1}{n}.
\end{aligned}
$$

for all $n \geq 1$.

Therefore, if n_0 is an integer for which $n_0 > \frac{1}{\epsilon}$, then

$$
\left|x_n + \frac{2}{3}\right| < \frac{1}{n} \leq \frac{1}{n_0} < \epsilon.
$$

whenever $n \geq n_0$, as desired.

Example 5

Determine whether following sequences are convergent or divergent.

(a) $\{n\}$

(b) $\{2^n\}$

(c) $\{(-1)^n\}$

(d) $\{\sin(\frac{n\pi}{2})\}_{n\geq 1}$

(a) The sequence $\{n\}$ diverges because no matter what p and ϵ we choose, the inequality $|n-p|<\epsilon$ can hold only for finitely many n.

Extra

Suppose, $\epsilon = 4$ and $p = 10$ then

$$\begin{aligned} |n-p| &< \epsilon \\ |n-10| &< 4 \\ -4 < n-10 &< 4 \end{aligned}$$

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$n-p$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5

(b) The sequence $\{2^n\}$ diverges because no matter what p and ϵ we choose, the inequality $|2^n - p| < \epsilon$ can hold only for finitely many n.

(c) The sequence defined by $\{(-1)^n\}$ is $\{-1, 1, -1, 1, \cdots\}$, and this sequence diverges by oscillation because the n^{th} term is always either 1 or -1. Thus $\{x_n\}$ cannot approach any one specific number p as n grows large.

(d) The sequence $\{\sin(\frac{n\pi}{2})\}_{n\geq 1}$ diverges because the sequence is $\{1, 0, -1, 0, 1, 0, \cdots\}$ and hence it does not converge to any number, by the same reasoning as above.

Null sequences

- A sequence $\{x_n\}$ that converges to zero is called a **null sequence**.
- For example, the sequence $\{\frac{1}{n}\}$ is a null sequence because it converges to zero.

Definition

Null sequence

A sequence $\{x_n\}$ is a **null sequence** if; for each positive number ϵ, there is a number n_0 such that $|x_n| < \epsilon$, for all $n \geq n_0$.

Example 1

Prove that the sequence $\left\{\frac{1}{n^3}\right\}$ is a null sequence.

We have to prove that for each positive number ϵ, there is a number n_0 such that

$$\left|\frac{1}{n^3}\right| < \epsilon, \quad \text{for all } n \geq n_0. \tag{1}$$

In order to find a suitable value of n_0 for (1) to hold, we rewrite the inequality $\left|\frac{1}{n^3}\right| < \epsilon$ in various equivalent ways until we spy a value for n_0 that will suit our purpose. Now

$$\begin{aligned}
\left|\frac{1}{n^3}\right| < \epsilon \quad &\Leftrightarrow \quad \frac{1}{n^3} < \epsilon \\
&\Leftrightarrow \quad n^3 > \frac{1}{\epsilon} \\
&\Leftrightarrow \quad n > \frac{1}{\sqrt[3]{\epsilon}}
\end{aligned}$$

Example 1
Solution ⇒ Cont...

If $\epsilon = 0.4$ then $\frac{1}{\sqrt[3]{\epsilon}} = 1.3572 \Rightarrow n \geq n_0 = 2$

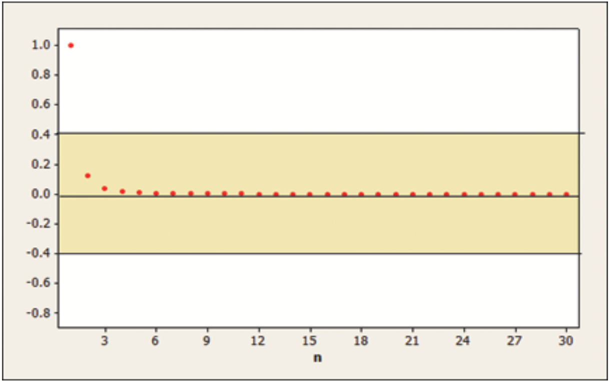


Figure: Sequence diagram

Example 1
Solution ⇒ Cont...

If $\epsilon = 0.2$ then $\frac{1}{\sqrt[3]{\epsilon}} = 1.7099 \Rightarrow n \geq n_0 = 2$

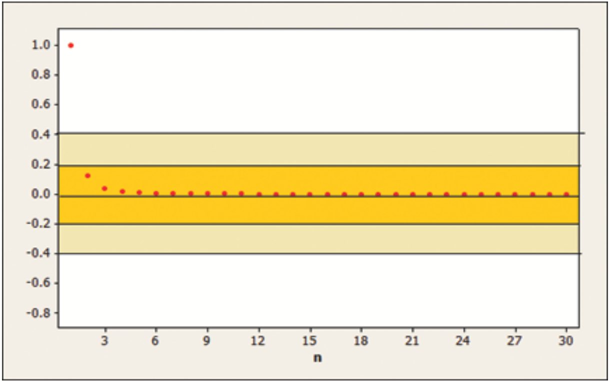


Figure: Sequence diagram

Example 1

Solution $\Rightarrow$ Cont...

If $\epsilon = 0.1$ then $\frac{1}{\sqrt[3]{\epsilon}} = 2.1543 \Rightarrow n \geq n_0 = 3$

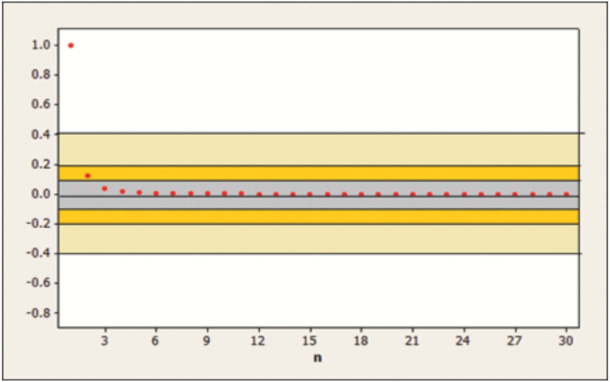


Figure: Sequence diagram

So, let us choose n_0 to be $\dfrac{1}{\sqrt[3]{\epsilon}}$.

With this choice of n_0, the above chain of equivalent inequalities shows us that, if $n \geq n_0$, then $\left|\frac{1}{n^3}\right| < \epsilon$.

Thus, with this choise of n_0, (1) holds; so $\{\frac{1}{n^3}\}$ is indeed null.

Example 2

Prove that the following sequence is not null

$$x_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

To prove that the sequence is not null, we have to show that it does not satisfy the definition.

In other words, we must show that the following statement is not true:

For each positive number ϵ, there is a number n_0 such that $|x_n| < \epsilon$, for all $n \geq n_0$.

So, what we have to show is that the following is true:

For some positive number ϵ, whatever n_0 one chooses $|x_n| \not< \epsilon$, for all $n \geq n_0$.

It can also be written as:

There is some positive number ϵ, such that whatever n_0 one chooses $|x_n| \not< \epsilon$, for all $n \geq n_0$.

So, we need to find some positive number ϵ with such a property.

The sequence diagram provides the clue.

Example 2

Solution ⇒ Cont...

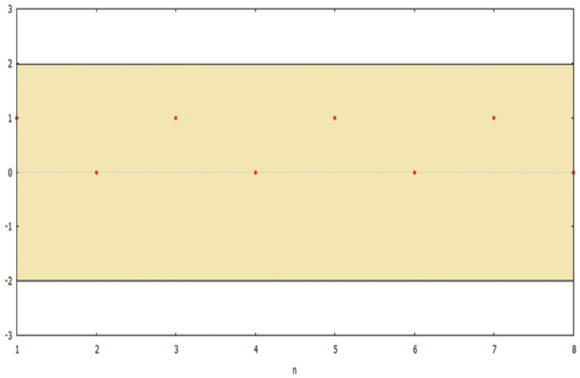


Figure: Sequence diagram

Example 2
Solution ⇒ Cont...

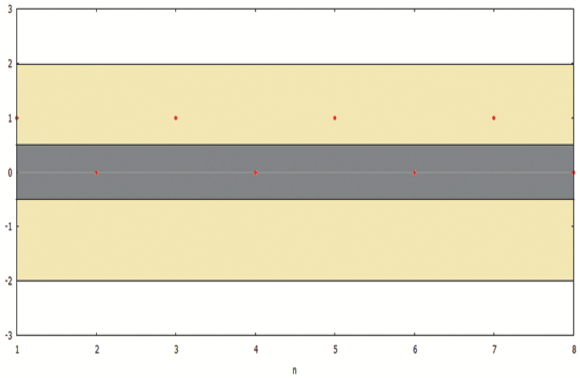


Figure: Strip of half-width

Example 2
Solution $\Rightarrow$ Cont...

If we choose $\epsilon = \frac{1}{2}$, then the point (n, x_n) lies outside the strip $(-\frac{1}{2}, \frac{1}{2})$ for every odd n.

In other words, whatever, n_0 one chooses, the statement

$$|x_n| < \epsilon, \quad \text{for all } n \geq n_0,$$

is false.

It follows that the sequence is not a null sequence.

- There is nothing special about the number $\frac{1}{2}$ here.
- Any positive value of ϵ less than 1 will serve for our purpose here.

Strategy for using the definition of null sequence

1. To show that $\{x_n\}$ is null, solve the inequality $|x_n| < \epsilon$ to find a number n_0 such that $|x_n| < \epsilon$, for all $n \geq n_0$.
2. To show that $\{x_n\}$ is not null, find one value of ϵ for which there is no number n_0 such that $|x_n| < \epsilon$, for all $n \geq n_0$.

Combination Rules

We now look at number of Rules for generating new sequences from old.

1. Power Rule
2. Sum Rule
3. Multiple Rule
4. Product Rule

Power Rule

If $\{x_n\}$ is a null sequence, where $x_n \geq 0$, for $n = 1, 2, \cdots$, and if $p > 0$, then $\{x_n^p\}$ is a null sequence.

We want to prove that $\{x_n^p\}$ is a null sequence; that is for each positive number ϵ, there is a number n_0 such that

$$x_n^p < \epsilon, \quad \text{for all } n \geq n_0. \tag{2}$$

We know that $\{x_n\}$ is null, so there is some number n_0 such that

$$x_n < \epsilon^{\frac{1}{p}}, \quad \text{for all } n \geq n_0. \tag{3}$$

Taking the p^{th} power of both sides of (3), we obtain the desired result (2), with the same value n_0.

Use the Power Rule to show that the following sequences are null:

(a) $\left\{\frac{1}{n^3}\right\}$

(b) $\left\{\frac{1}{\sqrt{n}}\right\}$

(c) $\left\{\frac{1}{\sqrt[5]{n}}\right\}$

(d) $\left\{\frac{1}{n^{\sqrt{7}}}\right\}$

We simply apply the Power Rule to the sequence $\{\frac{1}{n}\}$ that we saw earlier to be null, using the following positive powers:

(a) $p = 3$

(b) $p = \frac{1}{2}$

(c) $p = \frac{1}{5}$

(d) $p = \sqrt{7}$

So, according to the Power Rule, all the given sequences are null with above p values.

Sum Rule

If $\{x_n\}$ and $\{y_n\}$ are null sequences, then $\{x_n + y_n\}$ is a null sequence.

Sum Rule
Proof

We want to prove that the sum $\{x_n + y_n\}$ is null, that is for each positive number ϵ, there is a number n_0 such that

$$|x_n + y_n| < \epsilon, \quad \text{for all } n \geq n_0. \tag{4}$$

We know that $\{x_n\}$ and $\{y_n\}$ are null, so there are numbers n_1 and n_2 such that

$$\begin{aligned} |x_n| &< \frac{1}{2}\epsilon, \quad \text{for all } n \geq n_1, \text{ and} \\ |y_n| &< \frac{1}{2}\epsilon, \quad \text{for all } n \geq n_2. \end{aligned}$$

Hence, if $n_0 = \max(n_1, n_2)$, then both the previous inequalities holds, so if we add them we obtain, by the triangle inequality, that

$$|x_n + y_n| \leq |x_n| + |y_n| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon, \text{ for all } n \geq n_0.$$

Thus inequality (4) holds, with this value of n_0.

Use the Sum Rule to show that following sequences are null:

(a) $\left\{\frac{1}{n}+\frac{1}{n^3}\right\}$

(b) $\left\{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt[5]{n}}\right\}$

We may use known examples of null sequence to verify that the above sequences are also null. So,

(a) we can say that the sequence $\left\{\frac{1}{n}+\frac{1}{n^3}\right\}$ is null by applying the Sum Rule to the null sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{n^3}\right\}$.

(b) we can say that the sequence $\left\{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt[5]{n}}\right\}$ is null by applying the Sum Rule to the null sequences $\left\{\frac{1}{\sqrt{n}}\right\}$ and $\left\{\frac{1}{\sqrt[5]{n}}\right\}$.

Multiple Rule

If $\{x_n\}$ is a null sequence, then $\{\lambda x_n\}$ is a null sequence for any real number λ.

We want to prove that the multiple $\{\lambda x_n\}$ is null, that is for each positive number ϵ, there is a number n_0 such that

$$|\lambda x_n| < \epsilon, \quad \text{for all } n \geq n_0. \tag{5}$$

If $\lambda = 0$, this is obvious, and so we may assume that $\lambda \neq 0$.
We know that $\{x_n\}$ is null, so there is some number n_0 such that

$$|x_n| < \frac{1}{|\lambda|}\epsilon, \quad \text{for all } n \geq n_0.$$

Multiplying both sides of this inequality by the positive number $|\lambda|$, this gives us that

$$|\lambda x_n| < \epsilon, \quad \text{for all } n \geq n_0.$$

Thus the desired result (5) holds.

Use the Multiple Rule to show that the following sequences are null:

(a) $\left\{\frac{39\pi}{n^3}\right\}$

(b) $\left\{\frac{1}{(\sqrt{2}+\log\sqrt{5})\sqrt[5]{n}}\right\}$

We may use known examples of null sequence to verify that the above sequences are also null. So,

(a) we can say that the sequence $\left\{\frac{39\pi}{n^3}\right\}$ is null by applying the Multiple Rule to the null sequences $\left\{\frac{1}{n^3}\right\}$ with $\lambda = 39\pi$.

(b) we can say that sequence $\left\{\frac{1}{(\sqrt{2}+\log\sqrt{5})\sqrt[5]{n}}\right\}$ is null by applying the Multiple Rule to the null sequences $\dfrac{1}{\sqrt[5]{n}}$ with $\lambda = \frac{1}{(\sqrt{2}+\log\sqrt{5})}$.

Product Rule

If $\{x_n\}$ and $\{y_n\}$ are null sequences, then $\{x_n y_n\}$ is a null sequence.

We want to prove that the product $\{x_n y_n\}$ is null, that is for each positive number ϵ, there is a number n_0 such that

$$|x_n y_n| < \epsilon, \quad \text{for all } n \geq n_0. \tag{6}$$

We know that $\{x_n\}$ and $\{y_n\}$ are null, so there are numbers n_1 and n_2 such that

$$\begin{aligned} |x_n| &< \sqrt{\epsilon}, \quad \text{for all } n \geq n_1, \text{ and} \\ |y_n| &< \sqrt{\epsilon}, \quad \text{for all } n \geq n_2. \end{aligned}$$

Hence, if $n_0 = \max(n_1, n_2)$, then both two previous inequalities hold; so if we multiply them we obtain that

$$|x_n y_n| = |x_n| \times |y_n| < \sqrt{\epsilon} \times \sqrt{\epsilon} = \epsilon, \text{ for all } n \geq n_0.$$

Thus inequality (6) holds with this value of n_0.

Use the Product Rule to show that the following sequences are null:

(a) $\left\{ \dfrac{1}{n^3(2n-1)} \right\}$

(b) $\left\{ \dfrac{1}{n^4 \sqrt[5]{n}} \right\}$

We may use known examples of null sequence to verify that the above sequences are also null. So,

(a) we can say that the sequence $\left\{\frac{1}{n^3(2n-1)}\right\}$ is null by applying the Product Rule to the null sequences $\left\{\frac{1}{n^3}\right\}$ and $\left\{\frac{1}{(2n-1)}\right\}$.

(b) we can say that sequence $\left\{\frac{1}{n^4\sqrt[5]{n}}\right\}$ is null by applying the Product Rule to the null sequences $\left\{\frac{1}{n^4}\right\}$ and $\left\{\frac{1}{\sqrt[5]{n}}\right\}$.

Use the Combination Rules to show that the following sequences are null:

(a) $\left\{\dfrac{1}{(2n-1)^5}\right\}$

(b) $\left\{\dfrac{3}{\sqrt[7]{n}}+\dfrac{7}{(2n-1)^9}\right\}$

(c) $\left\{\dfrac{1}{5n^7(2n-1)^{\frac{1}{6}}}\right\}$

Basic null sequences

The following sequences are null sequences:

(a) $\left\{\dfrac{1}{n^p}\right\}$, for $p > 0$

(b) $\{c^n\}$, for $|c| < 1$

(c) $\{n^p c^n\}$, for $p > 0$, $|c| < 1$

(d) $\left\{\dfrac{c^n}{n!}\right\}$, for any real c

(e) $\left\{\dfrac{n^p}{n!}\right\}$, for $p > 0$

Basic null sequences
Examples

(a) $\left\{\dfrac{1}{n^p}\right\}$, for $p > 0 \Rightarrow \left\{\dfrac{1}{n^{10}}\right\}$

(b) $\{c^n\}$, for $|c| < 1 \Rightarrow \{(0.9)^n\}$

(c) $\{n^p c^n\}$, for $p > 0$, $|c| < 1 \Rightarrow \{n^3(0.9)^n\}$

(d) $\left\{\dfrac{c^n}{n!}\right\}$, for any real $c \Rightarrow \left\{\dfrac{10^n}{n!}\right\}$

(e) $\left\{\dfrac{n^p}{n!}\right\}$, for $p > 0 \Rightarrow \left\{\dfrac{n^{10}}{n!}\right\}$

Squeeze Rule

If $\{y_n\}$ is a null sequence and

$$|x_n| \leq y_n, \quad \text{for } n = 1, 2, \cdots ,$$

then $\{x_n\}$ is a null sequence.

We want to prove that $\{x_n\}$ is null; that is for each positive number ϵ, there is a number n_0 such that

$$|x_n| < \epsilon, \quad \text{for all } n \geq n_0. \tag{7}$$

We know that $\{y_n\}$ is null, so there is some number n_0 such that

$$|y_n| < \epsilon, \quad \text{for all } n \geq n_0. \tag{8}$$

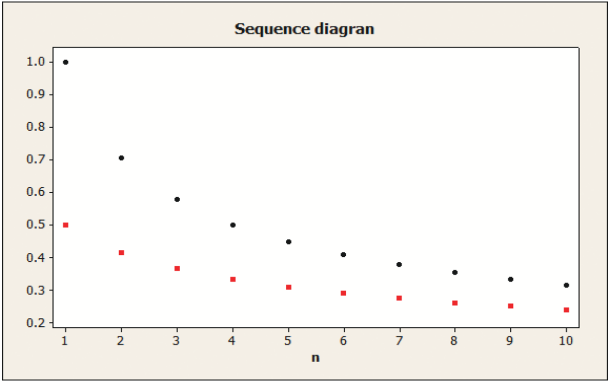
We also know that $|x_n| \leq y_n$, for $n = 1, 2, \cdots$, and hence it follows from (8) that

$$|x_n| \; (< |y_n|) < \epsilon, \quad \text{for all } n \geq n_0.$$

Thus inequality (7) holds, as required.

Squeeze Rule
Example

To illustrate this rule, we look at the sequence diagrams of the two sequences $\left\{\frac{1}{\sqrt{n}}\right\}$ and $\left\{\frac{1}{1+\sqrt{n}}\right\}$.



Since the $\left\{\frac{1}{\sqrt{n}}\right\}$ is null and

$$\frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}},$$

it follows from the Squeeze Rule that $\left\{\frac{1}{1+\sqrt{n}}\right\}$ is also a null sequence.

The trick in using the Squeeze Rule to prove that a given sequence $\{x_n\}$ is null is to think of a suitable sequence $\{y_n\}$ that dominates $\{x_n\}$ and is itself null.

Combination Rules for any convergent sequence

- So far we discussed Combination Rules and some other Rules only for null sequences.
- Fortunately, those rules can easily be extended for any other convergent sequence as well.

Combination Rules for any convergent sequence
Cont...

If $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} y_n = q$, then

Sum Rule $\lim_{n\to\infty}(x_n + y_n) = p + q$

Multiple Rule $\lim_{n\to\infty}(\lambda x_n) = \lambda p$, for any real number λ

Product Rule $\lim_{n\to\infty}(x_n y_n) = pq$

Quotient Rule $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{p}{q}$, provided that $q \neq 0$

Reciprocal Rule $\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = \frac{1}{p}$, provided that $p \neq 0$

Proofs of Combination Rules

We prove the Sum Rule, the Multiple Rule, and the Product Rule by using the corresponding Combination Rules for null sequences.

In that case

$$\lim_{n\to\infty} x_n = p$$

means that $\{x_n - p\}$ is null.

- We have already shown that $x_n = \frac{n+1}{n}$ is a sequence which converges to $p = 1$.
- If we consider $x_n - p$ then we get $x_n - p = \frac{n+1}{n} - 1 = \frac{1}{n}$.
- We know that $\frac{1}{n}$ is a null sequence.
- It indicates that if we consider $x_n - p$, the resulting sequence $\{x_n - p\}$ will be a null sequence.

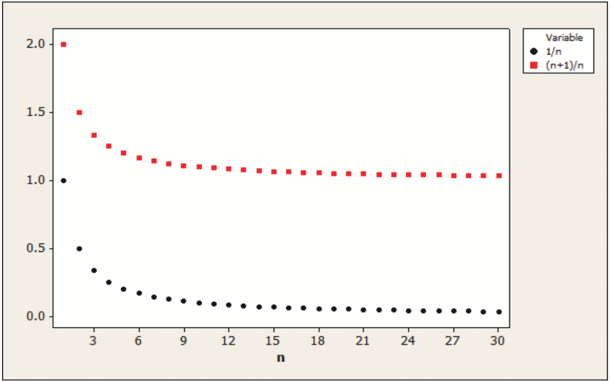
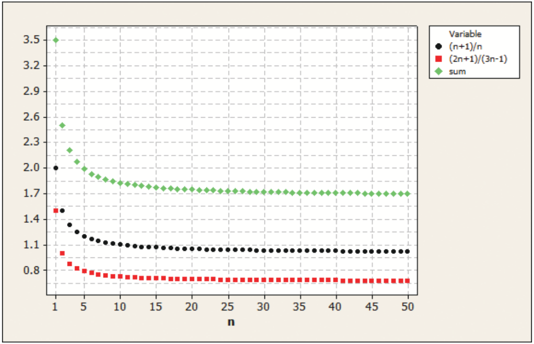


Figure: Sequence diagrams for $\frac{n+1}{n}$ and $\frac{1}{n}$.

Sum Rule
Example

If $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lim_{n\to\infty} \frac{(2n+1)}{(3n-1)} = \frac{2}{3}$, then

$$\lim_{n\to\infty}\left(\frac{n+1}{n} + \frac{(2n+1)}{(3n-1)}\right) = 1 + \frac{2}{3} = 1.66667$$



Sum Rule
Proof

$\{x_n - p\}$ and $\{y_n - q\}$ are null sequences. Since

$$(x_n + y_n) - (p + q) = (x_n - p) + (y_n - q)$$

we deduce that $\{(x_n + y_n) - (p + q)\}$ is null, by the Sum Rule for null sequences as follows:

$$\begin{aligned}
\lim_{n\to\infty}((x_n + y_n) - (p + q)) &= \lim_{n\to\infty}(x_n - p) + \lim_{n\to\infty}(y_n - q) \\
\lim_{n\to\infty}((x_n + y_n) - (p + q)) &= 0 + 0 = 0
\end{aligned}$$

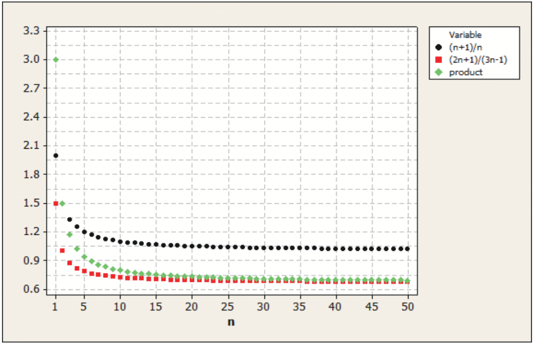
It implies that

$$\lim_{n\to\infty}(x_n + y_n) = p + q.$$

Product Rule
Example

If $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lim_{n\to\infty} \frac{(2n+1)}{(3n-1)} = \frac{2}{3}$, then

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\times\frac{(2n+1)}{(3n-1)}\right) = 1\cdot\frac{2}{3} = \frac{2}{3}$$



The idea here is to express $x_n y_n - pq$ in terms of $x_n - p$ and $y_n - q$

$$x_n y_n - pq = (x_n - p)(y_n - q) + q(x_n - p) + p(y_n - q).$$

Since $\{x_n - p\}$ and $\{y_n - q\}$ are null, we deduce that $\{x_n y_n - pq\}$ is null, by the Combination Rules for null sequences as follows:

$$\begin{aligned}\lim_{n\to\infty}(x_n y_n - pq) &= \lim_{n\to\infty}(x_n - p)(y_n - q) + \lim_{n\to\infty} q(x_n - p)\\ &\quad + \lim_{n\to\infty} p(y_n - q)\\ &= 0 + 0 + 0 = 0.\end{aligned}$$

It implies that

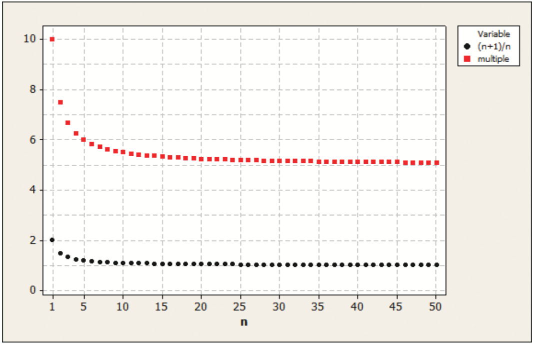
$$\lim_{n\to\infty} x_n y_n = pq.$$

Multiple Rule

Example

If $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lambda = 5$ then

$$\lim_{n\to\infty} \lambda \left(\frac{n+1}{n} \right) = 5 \times 1 = 5.$$

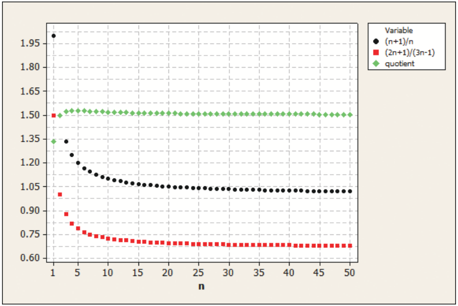


Note that the Multiple Rule is a just a special case of the Prodcut Rule in which the sequence $\{y_n\}$ is a constant sequence.

Quotient Rule
Example

If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{(2n+1)}{(3n-1)} = \frac{2}{3}$, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$



To prove the Quotient Rule, we need to use the following lemma.

Quotient Rule

Lemma 1

If $\lim_{n\to\infty} x_n = p$ and $p > 0$, then there is a number n_0 such that $x_n > \frac{1}{2}p$, for all $n \geq n_0$.

Since $\lim_{n\to\infty} x_n = p$, by the definition of convergence:

$$|x_n - p| < \epsilon, \quad \text{for all } n \geq n_0.$$

Since any positive value can be taken as ϵ, we take $\epsilon = \frac{1}{2}p$, then:

$$\begin{aligned} |x_n - p| &< \frac{1}{2}p, \quad \text{for all } n \geq n_0 \\ -\frac{1}{2}p < x_n - p &< \frac{1}{2}p, \quad \text{for all } n \geq n_0 \\ \frac{1}{2}p < x_n &< \frac{3}{2}p, \quad \text{for all } n \geq n_0. \end{aligned}$$

So the left-hand inequality gives

$$x_n > \frac{1}{2}p, \quad \text{for all } n \geq n_0, \text{ as required.}$$

We assume that $q > 0$; the proof for the case $q < 0$ is similar.

Once again the idea is to write the required expression in terms of $x_n - p$ and $y_n - q$:

$$\frac{x_n}{y_n} - \frac{p}{q} = \frac{x_n q - y_n p}{y_n q} = \frac{q(x_n - p) - p(y_n - q)}{y_n q}.$$

$q(x_n - p) - p(y_n - q)$ is certainly a null sequence, but the denominator is rather awkward.

Some of the terms y_n may take value 0, in which case the expression is undefined.

However, by Lemma 1, we know that for some n_0 we have

$$y_n > \frac{1}{2}q, \quad \text{for all } n \geq n_0.$$

Thus for all $n \geq n_0$:

$$\begin{aligned}
\left|\frac{x_n}{y_n} - \frac{p}{q}\right| &= \frac{|q(x_n - p) - p(y_n - q)|}{y_n q} \\
&\leq \frac{|q(x_n - p) - p(y_n - q)|}{\frac{1}{2}q^2} \\
&\leq \frac{|q| \times |(x_n - p)| + |p| \times |(y_n - q)|}{\frac{1}{2}q^2}
\end{aligned}$$

Since this last expression defines a null sequence, it follows by Squeeze Rule that $\left\{\frac{x_n}{y_n} - \frac{p}{q}\right\}$ is null.

Applying Combination Rules

- So far we tested the convergence of a given sequence $\{x_n\}$ when we know the value of p in advance.
- But usually, we are not given the value of p.
- We are only given a sequence $\{x_n\}$ and asked to decide whether or not it coverges and, if it does find its limt.
- Combination Rules can be used to deal with such cases.

Example

Show that each of the following sequences $\{x_n\}$ is convergent, and find its limits:

(a) $x_n = \dfrac{(2n+1)(n+2)}{3n^2+3n}$

(b) $x_n = \dfrac{2n^2+10^n}{n!+3n^3}$

(c) $x_n = \dfrac{n^3+2n^2+3}{2n^3+1}$

(d) $x_n = \dfrac{n!+(-1)^n}{2^n+3n!}$

Example
Solution

(a) In this case we divide both the numerator and denominator by n^2 to give

$$\begin{aligned} x_n &= \frac{(2n+1)(n+2)}{3n^2+3n} \\ &= \frac{\left(2+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{3+\frac{3}{n}} \end{aligned}$$

Since $\{\frac{1}{n}\}$ is a basic null sequence, we find by the Combination Rules that

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{\left(2+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{3+\frac{3}{n}} \\ \lim_{n\to\infty} x_n &= \frac{(2+0)(1+0)}{3+0} = \frac{2}{3}. \end{aligned}$$

Example

Solution $\Rightarrow$ Cont....

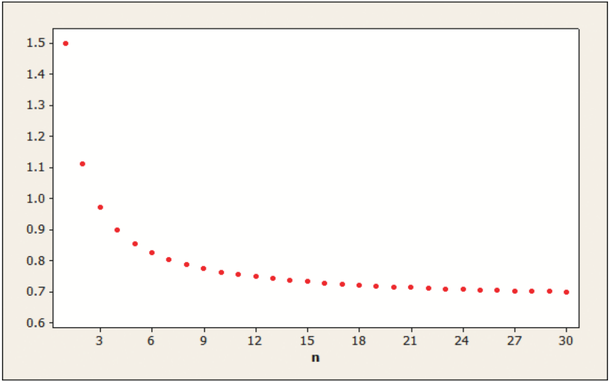


Figure: Sequence diagram for $x_n = \frac{(2n+1)(n+2)}{3n^2+3n}$

(b) This time we divide both the numerator and denominator by $n!$ to give

$$x_n = \frac{2n^2 + 10^n}{n! + 3n^3} = \frac{\frac{2n^2}{n!} + \frac{10^n}{n!}}{1 + \frac{3n^3}{n!}}$$

Since $\{\frac{n^2}{n!}\}$, $\{\frac{10^n}{n!}\}$ and $\{\frac{n^3}{n!}\}$ are all basic null sequences, we find by Combination Rules that

$$\lim_{n\to\infty} = \frac{0+0}{1+0} = 0.$$

(c) In this case we divide both the numerator and denominator by n^3 to give

$$\begin{aligned} x_n &= \frac{n^3 + 2n^2 + 3}{2n^3 + 1} \\ &= \frac{1 + \frac{2}{n} + \frac{3}{n^3}}{2 + \frac{1}{n^3}} \end{aligned}$$

Since $\{\frac{1}{n}\}$ and $\{\frac{1}{n^3}\}$ are basic null sequences, we find by the Combination Rules that

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{3}{n^3}}{2 + \frac{1}{n^3}} \\ &= \frac{1+0+0}{2+0} = \frac{1}{2} \end{aligned}$$

Example

Solution $\Rightarrow$ Cont....

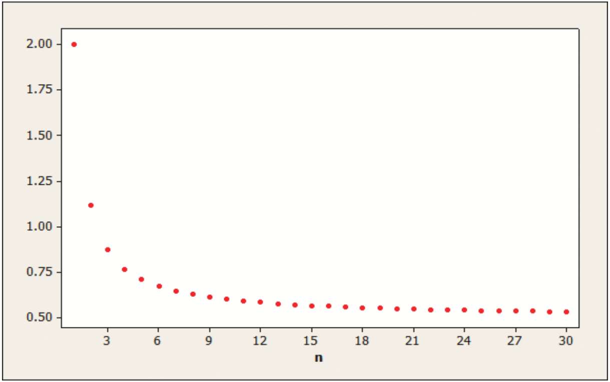


Figure: Sequence diagram for $x_n = \frac{n^3+2n^2+3}{2n^3+1}$

(d) This time we divide both the numerator and denominator by $n!$ to give

$$\begin{aligned} x_n &= \frac{n! + (-1)^n}{2^n + 3n!} \\ &= \frac{1 + \frac{(-1)^n}{n!}}{\frac{2^n}{n!} + 3} \end{aligned}$$

Since $\{\frac{(-1)^n}{n!}\}$ and $\{\frac{2^n}{n!}\}$ are all basic null sequences, we find by Combination Rules that

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{1 + \frac{(-1)^n}{n!}}{\frac{2^n}{n!} + 3} \\ &= \frac{1+0}{0+3} = \frac{1}{3}. \end{aligned}$$

General version of Squeeze Rule

If

1. $x_n \leq y_n \leq z_n$, for $n = 1, 2, \cdots,$
2. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = p$,

then $\lim_{n\to\infty} y_n = p$.

General version of Squeeze Rule
Proof

By the combination Rules

$$\lim_{n\to\infty}(z_n - x_n) = p - p = 0,$$

so that $\{z_n - x_n\}$ is a null sequence. Also, by condition 1

$$0 \le y_n - x_n \le z_n - x_n, \quad \text{for } n = 1, 2, \cdots$$

and so $\{y_n - x_n\}$ is null, by Squeeze Rule for null sequences. Now we write y_n in the form

$$y_n = (y_n - x_n) + x_n.$$

Hence by the Combination Rules

$$\begin{aligned} \lim_{n\to\infty} y_n &= \lim_{n\to\infty}(y_n - x_n) + \lim_{n\to\infty} x_n \\ &= 0 + p = p. \end{aligned}$$

Limit Inequality Rule

If $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} y_n = q$, and also

$$x_n \leq y_n, \quad \text{for all } n = 1, 2, \cdots$$

then $p \leq q$.

Limit Inequality Rule
Proof

Suppose that $x_n \to p$ and $y_n \to q$, where $x_n \leq y_n$ for $n = 1, 2, \cdots$, but that it is not true that $p \leq q$. Then $p > q$ and so, by the Combination Rules

$$\lim_{n\to\infty}(x_n - y_n) = p - q > 0.$$

Hence, by Lemma 1, there is an n_0 such that

$$x_n - y_n > \frac{1}{2}(p - q), \quad \text{for all } n \geq n_0. \tag{9}$$

However, we assumed that $x_n - y_n \leq 0$, for $n = 1, 2, \cdots$, so statement (9) is a contradiction.

Hence it is true that $p \leq q$.

Corollary 1

If $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} x_n = q$, then $p = q$.

Applying Limit Inequality Rule with $x_n = y_n$, we deduce that $p \leq q$ and $q \leq p$.

Hence $p = q$.

Theorem 1

If $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} |x_n| = |p|$.

Using the reverse form of the Triangle Inequality, we obtain

$$||x_n| - |p|| \leq |x_n - p|, \quad \text{for } n = 1, 2, \cdots .$$

Since $\{x_n - p\}$ is null, we deduce from the Squeeze Rule for null sequences that $\{|x_n| - |p|\}$ is null, as required.

Types of divergent sequences

- A sequence is divergent if it is not convergent.
- We now investigate the behavior of sequences which do not convergent.
- Each of the below sequences is divergent but, as you can see, they behave differently.

Types of divergent sequences
Cont...

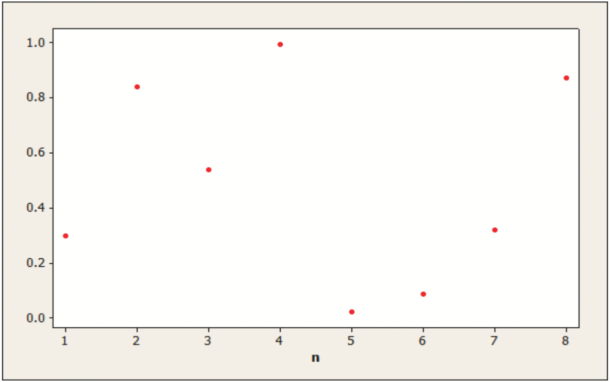


Figure: Sequence diagram of $\{4(1-x_n)x_n\}$ where $x_1 = 0.3$.

Types of divergent sequences
Cont...

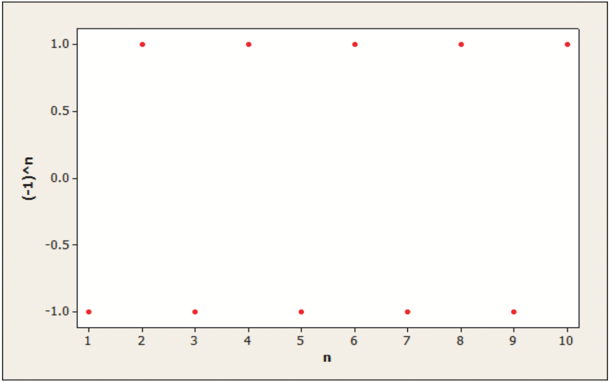


Figure: Sequence diagram of $\{(-1)^n\}$

Types of divergent sequences
Cont...

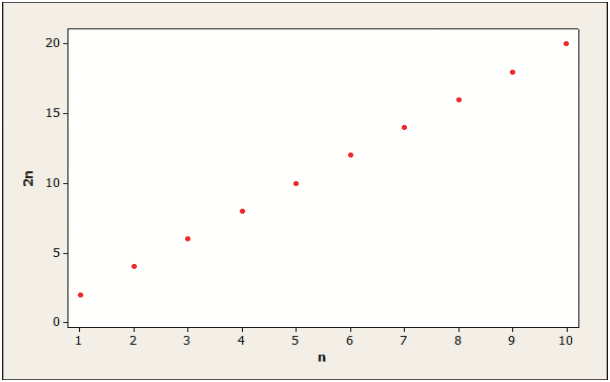


Figure: Sequence diagram of $\{2n\}$.

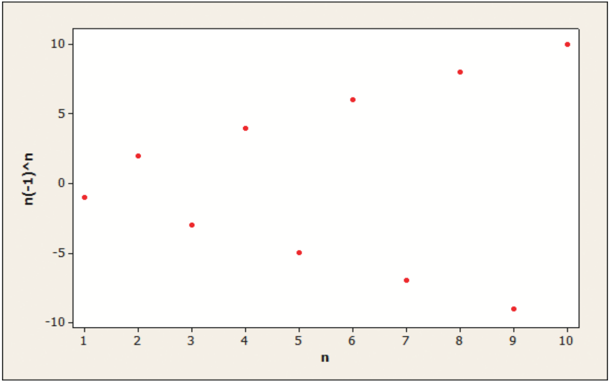


Figure: Sequence diagram of $\{n(-1)^n\}$.

Criteria for divergence

- In here our intention is to obtain criteria for divergence, which avoid having to argue directly from the definition.
- We obtain two criteria by establishing certain properties, which are necesarily possessed by a convergent sequence; if a sequence does not have these properties, then it must be divergent.

- One such property possessed by a convergent sequence is that it must be bounded.
- The idea of subsequences which we will discuss later can also be used for establishing certain properties for convergent sequences.
- First, we will talk about boundedness.

Theorem 2
Boundedness Theorem

If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.

We know that $x_n \to p$, for some real number p. Thus $\{x_n - p\}$ is a null sequnce, and so there is a number n_0 such that

$$|x_n - p| < \epsilon, \text{ for all } n \geq n_0.$$

By taking $\epsilon = 1$ in the defintion of a null sequence, we have:

$$|x_n - p| < 1, \text{ for all } n \geq n_0.$$

For simplicity in the rest of the proof, we shall now assume that our initial choice of n_0 is a positive integer. Now

$$\begin{aligned} |x_n| &= |x_n - p + p| \\ &\leq |x_n - p| + |p|, \text{ by the Triangle Inquality.} \end{aligned}$$

It follows that

$$|x_n| \leq 1 + |p|, \quad \text{for all } n \geq n_0.$$

This is the type of inequality needed to prove that $\{x_n\}$ is bounded, but it does not include the terms $x_1, x_2, \cdots, x_{n_0}$.

To complete the proof, we let M be the maximum of the numbers $|x_1|, |x_2|, \cdots, |a_{n_0}|, 1 + |p|$.

It follows that

$$|x_n| \leq M, \text{ for } n = 1, 2, \cdots,$$

as required.

Corollary 2

If $\{x_n\}$ is unbounded, then $\{x_n\}$ is divergent.

Example 1

(a) The sequences $\{2n\}$ and $\{n^2\}$ are both unbounded, since, for each number M, we can find terms of these sequences whose absolute values are greater than M. So they are both divergent.

(b) The sequence $\{(-1)^n\}$ is bounded, because

$$|(-1)^n| \leq 1, \text{for } n = 1, 2, \cdots .$$

However it is not necessarily convergent.

Example 2

Classify the following sequences as convergent or divergent, and as bounded or unbounded:

(a) $\{\sqrt{n}\}$

(b) $\left\{\dfrac{n^2 + n}{n^2 + 1}\right\}$

(c) $\{(-1)^n n^2\}$

Example 2
Solution

(a) $\{\sqrt{n}\}$ is unbounded, and hence divergent by Corollary 2.

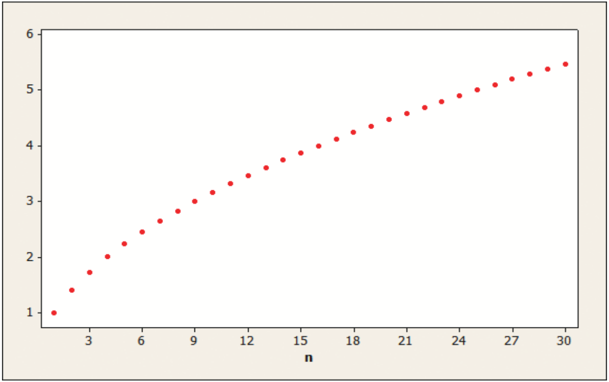


Figure: Sequence diagram of $\{\sqrt{n}\}$.

Example 2

Solution ⇒ Cont...

(b) $\{\frac{n^2+n}{n^2+1}\}$ is convergent with limit 1, and hence bounded, by Theorem 2. Infact

$$\frac{n^2+n}{n^2+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{n^2}} \leq 1 + \frac{1}{n} \leq 2, \quad \text{for } n = 1, 2, \cdots$$

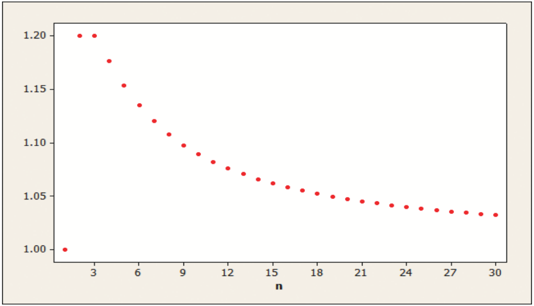


Figure: Sequence diagram of $\{\frac{n^2+n}{n^2+1}\}$.

(c) $\{(-1)^n n^2\}$ is unbounded, and hence divergent, by Corollary 2.

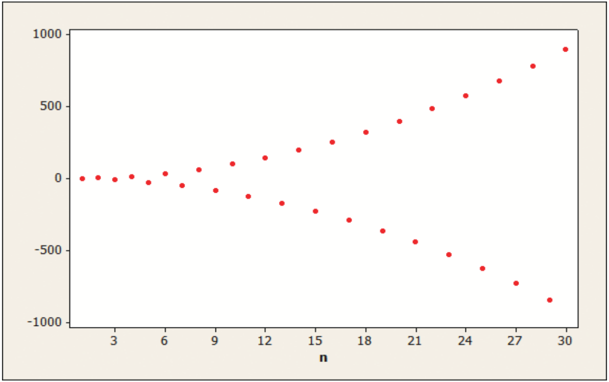


Figure: Sequence diagram of $\{(-1)^n n^2\}$.

The sequence $\{x_n\}$ tends to infinity if; for each positive number M, there is a number n_0 such that

$$x_n > M, \text{ for all } n > n_0.$$

In this case, we write

$$x_n \to \infty \text{ as } n \to \infty.$$

Definition

Sequences which tend to infinity $\Rightarrow$ Cont...

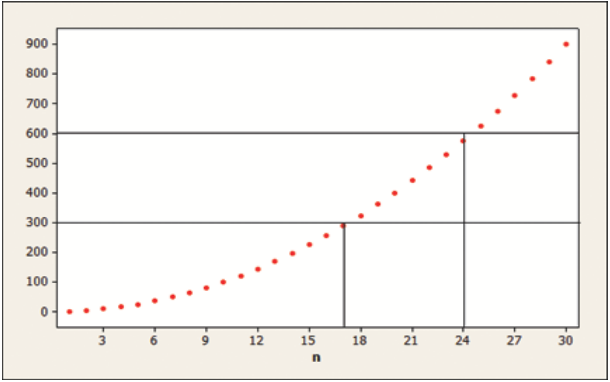


Figure: The sequence diagram of n^2.

Remark

- If a sequence tends to infinity, then it is unbounded and hence divergent, by Corollary 2.
- If a sequence tends to infinity, then this remains true if we add, delete or alter a finite number of terms.

Reciprocal Rule

- There is a version of the Reciprocal Rules for sequences which tends to infinity.
- This unables us to use our knowledge of null sequences to identify sequences which tend to infinity.

Theorem 3
Reciprocal Rule

(a) If the sequence $\{x_n\}$ satisfies both of the following conditions:

1. $\{x_n\}$ is eventually positive,
2. $\{\frac{1}{x_n}\}$ is a null sequence,

then $x_n \to \infty$.

(b) If $x_n \to \infty$, then $\frac{1}{x_n} \to 0$.

Example

Use the Reciprocal Rule to prove that the following sequences tend to infinity:

(a) $\left\{\dfrac{n^5}{3}\right\}$

(b) $\{n! + 2^n\}$

(c) $\{n! - 10^n\}$

(a) Each term of the sequence $\left\{\frac{n^5}{3}\right\}$ is positive and $\frac{1}{n^5/3} = \frac{3}{n^5}$. Now, $\left\{\frac{1}{n^5}\right\}$ is a basic null sequence and so $\frac{3}{n^5}$ is null, by the Multiple Rule. Hence $\left\{\frac{n^5}{3}\right\}$ tends to infinity, by the Reciprocal Rule.

(b) Each term of the sequence $\{n! + 2^n\}$ is positive and

$$\lim_{n\to\infty} \frac{1}{n! + 2^n} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{1 + \frac{2^n}{n!}} = \frac{0}{1+0} = 0,$$

by the Combination Rules. Hence $\{n! + 2^n\}$ tends to infinity, by the Reciprocal Rule.

(c) First note that

$$n! - 10^n = n!\left(1 - \frac{10^n}{n!}\right), \quad \text{for } n = 1, 2, \cdots$$

Since $\{\frac{10^n}{n!}\}$ is a basic null sequence, we know that $\frac{10^n}{n!}$ is eventually less than 1, and so $n! - 10^n$ is eventually positive. Also

$$\lim_{n\to\infty} \frac{1}{n! - 10^n} = \lim_{n\to\infty} \frac{\frac{1}{n!}}{1 - \frac{10^n}{n!}} = \frac{0}{1-0} = 0,$$

by the Combination Rules.

Hence $\{n! - 10^n\}$ tends to infinity, by the Reciprocal Rule.

Combination Rules and Squeeze Rule

There are also versions of the Combination Rules and Squeeze Rule for sequences which tend to infinity. We state these without proof.

Combination Rules

If $\{x_n\}$ tends to infinity and $\{y_n\}$ tends to infinity, then:

Sum Rule	$\{x_n + y_n\}$ tends to infinity;
Multiple Rule	$\{\lambda x_n\}$ tends infinity, for $\lambda > 0$;
Product Rule	$\{x_n y_n\}$ tends to infinity.

Example

For each of the following sequences $\{x_n\}$, prove that $x_n \to \infty$.

(a) $\left\{\dfrac{2^n}{n}\right\}$

(b) $\left\{\frac{2^n}{n} + 5n^{100}\right\}$

(a) Each term of $\{\frac{2^n}{n}\}$ is positive, and $\{\frac{n}{2^n}\}$ is a basic null sequence. Hence $\frac{2^n}{n} \to \infty$ by the Reciprocal Rule.

(b) Each term of $\{n^{100}\}$ is positive, and $\{\frac{1}{n^{100}}\}$ is a basic null sequence. Hence $n^{100} \to \infty$ by the Reciprocal Rule. By Multiple Rule $5n^{100} \to \infty$. From part (a) we know that $\frac{2^n}{n} \to \infty$. Hence by Sum Rule we have $\frac{2^n}{n} + 5n^{100} \to \infty$.

Squeeze Rule

If $\{y_n\}$ tends to infinity, and

$$x_n \geq y_n, \quad \text{for } n = 1, 2, \cdots ,$$

then $\{x_n\}$ tends to infinity.

Example

Show that the sequence $\left\{\frac{2^n}{n} + 5n^{100}\right\}$ tends to infinity.

Example
Solution

We know that $\frac{2^n}{n} \to \infty$, by part (a) of the above example, and that

$$\frac{2^n}{n} + 5n^{100} \geq \frac{2^n}{n}, \quad \text{for } n = 1, 2, \cdots$$

Hence $\dfrac{2^n}{n} + 5n^{100} \to \infty$, by the Squeeze Rule.

Definition

The sequence $\{x_n\}$ tends to minus infinity if

$$-x_n \to \infty \text{ as } n \to \infty.$$

In this case, we write

$$x_n \to -\infty \text{ as } n \to \infty.$$

Example

- The sequence $\{-n^2\}$ and $\{10^n - n!\}$ both tend to minus infinity, because $\{n^2\}$ and $\{n! - 10^n\}$ tend to infinity.
- Sequences which tend to minus infinity are unbounded, and hence divergent.

Subsequences

- Consider the bounded divergent sequence $\{(-1)^n\}$.
- This sequence splits naturally into two:
  1. the *even* terms $x_2, x_4, \cdots, x_{2k}, \cdots$, each of which equals 1;
  2. the *odd* terms $x_1, x_3, \cdots, x_{2k-1}, \cdots$, each of which equals -1;
- Both of these are sequence in thier own right, and we call them the **even subsequence** $\{x_{2k}\}$ and the **odd subsequence** $\{x_{2k-1}\}$.

Subsequences
Cont...

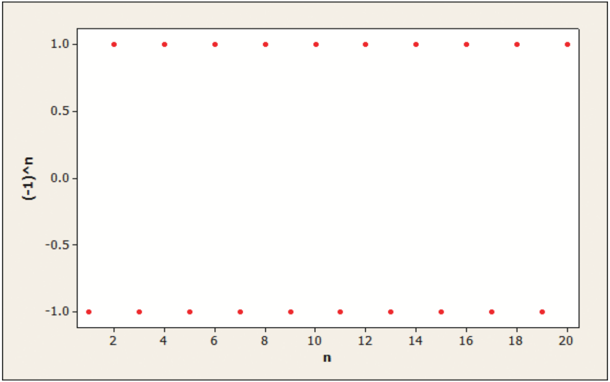


Figure: The sequence diagram of $(-1)^n$.

In general, given a sequence $\{x_n\}$ we may consider many different subsequences, such as

- $\{x_{3k}\}$, comprising the terms $x_3, x_6, x_9, \cdots$;
- $\{x_{4k+1}\}$, comprising the terms $x_5, x_9, x_{13}, \cdots$;
- $\{x_{2k!}\}$, comprising the terms $x_2, x_4, x_{12}, \cdots$.

The sequence $\{x_{n_k}\}$ is a **subsequence** of the sequence $\{x_n\}$ if $\{n_k\}$ is a strictly increasing sequence of positive integers; that is, if

$$n_1 < n_2 < n_3 < \cdots$$

Example

Let $x_n = n^2$, for $n = 1, 2, \cdots$. Write down the first five terms of each of the subsequences $\{x_{n_k}\}$, where:

(a) $n_k = 2k$;

(b) $n_k = 4k - 1$.

(a) $x_2 = 4,\ x_4 = 16,\ x_6 = 36,\ x_8 = 64,\ x_{10} = 100;$

(b) $x_3 = 9,\ x_7 = 49,\ x_{11} = 121,\ x_{15} = 225,\ x_{19} = 361.$

Inheritance property of subsequences

For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$:

1. If $x_n \to p$ as $n \to \infty$, then $x_{n_k} \to p$ as $k \to \infty$;
2. If $x_n \to \infty$ as $n \to \infty$, then $x_{n_k} \to \infty$ as $k \to \infty$.

First Subsequence Rule

The sequence $\{x_n\}$ is divergent if it has two convergent subsequences with different limits.

Example

- The sequence $\{(-1)^n\}$ has two convergent subsequences which have different limits.
- The even subsequence converges to 1 and the odd subsequence converges to -1.
- So, the sequence $\{(-1)^n\}$ is divergent, by the First Subsequence Rule.

Second Subsequence Rule

The sequence $\{x_n\}$ is divergent if it has a subsequence which tends to infinity or a subsequence which tends to minus infinity.

Example

- The sequence $\{n^{(-1)^n}\}$ has a subsequence (the even subsequence) which tends to infinity.
- So, $\{n^{(-1)^n}\}$ is divergent by the Second Subsequence Rule.

Theorem 4

If the odd and even subsequences of $\{x_n\}$ both tend to the same limit p, then

$$\lim_{n\to\infty} x_n = p.$$

Thank you !