Calculus (Real Analysis I)

(MAT122β)

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The Real Number System

Axioms

What is an axiom?

- An **axiom** is a statement that is assumed to be true without any proof.
- **Eg:** Let x and y be real numbers. Then $x + y$ and xy are also real numbers.
- Axioms are the principal building blocks of proving statements.
- They serve as the starting point of other mathematical statements.
- These statements, which are derived from axioms, are called **theorems**.

What is a theorem?

- A **theorem**, by definition, is a statement proven based on axioms, other theorems, and some set of logical connectives.
- Theorems are often proven through rigorous mathematical and logical reasoning, and the process towards the proof will, of course, involve one or more axioms and other statements which are already accepted to be true.
- Theorems are more often challenged than axioms, because they are subject to more interpretations, and various derivation methods.

What is a methematical proof?

- A **proof** is a deductive argument for a mathematical statement.
- In the argument, other previously established statements, such as theorems, can be used.
- A proof must demonstrate that a statement is always true.
- An unproven statement that is believed true is known as a **conjecture**.

Different type of numbers

Natural ($\mathbb{N}$)	1, 2, 3, 4, 5, 6, 7, ..., n
Positive integers	1, 2, 3, 4, 5, ..., n
Negative integers	-1, -2, -3, -4, -5, ..., -n
Integers ($\mathbb{Z}$)	-n, ..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ..., n
Rational ($\mathbb{Q}$)	A number that can be expressed as p/q where p and q are integers and q is not zero
Irrational ($\overline{\mathbb{Q}}$)	A number that cannot be expressed as p/q where p and q are integers and q is not zero
Real ($\mathbb{R}$)	A value that represents a quantity along a continuous line

Real numbers ($\mathbb{R}$)

- The real numbers include all of the measuring numbers.
- The set of real numbers includes all integers, all rational and the all irrational numbers.
- Every rational number is also a real number.
- It is not the case, however, that every real number is rational.

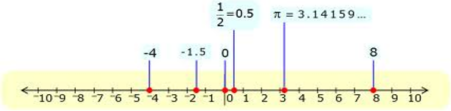
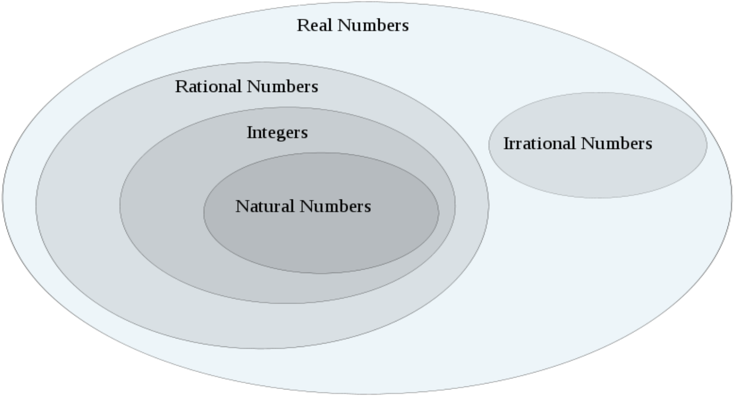


Diagram of number system



Axioms for real numbers

- Let $\mathbb{R}$ be the set of real numbers with two binary operations: addition $x + y$ and multiplication $x \cdot y$ or just xy.
- In a written expression involving both additions and multiplications, multiplications take precedence over addition.
- Furthermore, $\mathbb{R}$ obeys the following sets of axioms.

The field axioms describe the algebraic properties of real numbers.

A1: Closure property. For every $x, y \in \mathbb{R} \Rightarrow x + y \in \mathbb{R}$

A2: Associative property. For every $x, y, z \in \mathbb{R}$
$(x + y) + z = x + (y + z)$

A3: Identity element. For every $x \in \mathbb{R}$ we have $0 \in \mathbb{R}$ such that
$x + 0 = 0 + x = x$

A4: Inverse element. For every $x \in \mathbb{R}$ we have $-x \in \mathbb{R}$ such that
$x + (-x) = (-x) + x = 0$

A5: Commutative property. For every $x, y \in \mathbb{R}$
$x + y = y + x$

M1: Closure property. For every $x, y \in \mathbb{R} \Rightarrow xy \in \mathbb{R}$

M2: Associative property. For every $x, y, z \in \mathbb{R}$
$(xy)z = x(yz)$

M3: Identity element. For every $x \in \mathbb{R}$ we have $1 \in \mathbb{R}$ such that
$x \cdot 1 = 1 \cdot x = x$

M4: Inverse element. For every $x \in \mathbb{R}, x \neq 0$, $\exists x' \in \mathbb{R}$ such that
$xx' = x'x = 1$

M5: Commutative property. For every $x, y \in \mathbb{R}$
$xy = yx$

D: Distributive property. For every $x, y, z \in \mathbb{R}$
$x(y + z) = xy + xz$

- A **field** is a set together with two operations, usually called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the above axioms hold.
- Therefore the collection $(\mathbb{R}, +, \cdot)$ is a field.

Consequences of the algebric axioms

- We can define subtraction in terms of addition and aditive inverse by

$$a - b = a + (-b).$$

- Similarly if $b \neq 0$

$$a \div b = \frac{a}{b} = ab^{-1}.$$

Proposition 2.1

The additive and multiplicative identities are unique.

Proposition 2.1
Proof

Suppose that 0 and $0'$ are two additive identities on $\mathbb{R}$, clearly $0, 0' \in \mathbb{R}$.
If $0 \in \mathbb{R}$ and $0' \in \mathbb{R}$ is an identity we can write

$$0 + 0' = 0' + 0 = 0 \tag{1}$$

Similarly $0' \in \mathbb{R}$ and $0 \in \mathbb{R}$ is an identity

$$0' + 0 = 0 + 0' = 0' \tag{2}$$

From (1) and (2)

$$\begin{aligned} 0 = 0 + 0' &= 0' + 0 = 0' \\ 0 &= 0' \Rightarrow \text{Hence the proof.} \end{aligned}$$

The argument for the multiplicative identity is similar.

Proposition 2.2

For any given $x \in \mathbb{R}$ there is a unique additive inverse $(-x)$ and for any non zero $x \in \mathbb{R}$ there is a unique multiplicative inverse (x^{-1}).

Proposition 2.2
Proof

Let y_1, y_2 be two additive inverses of $x \in \mathbb{R}$

$$y_1 + x = x + y_1 = 0 = y_2 + x = x + y_2. \tag{3}$$

$$\begin{aligned} y_1 &= y_1 + 0 \\ &= y_1 + (x + y_2) \\ &= (y_1 + x) + y_2 \\ &= 0 + y_2 \quad (\text{by}(3)) \\ &= y_2 \Rightarrow \text{Hence the proof} \end{aligned}$$

Proposition 2.2
Proof⇒Cont...

Let y_1, y_2 be two multiplicative inverses of $x \in \mathbb{R}, x \neq 0$.

$$y_1x = xy_1 = 1 = y_2x = xy_2 \tag{4}$$

On the other hand we have

$$\begin{aligned} y_1 = y_1 \cdot 1 &= y_1(xy_2) \\ &= (y_1x)y_2 \\ &= 1 \cdot y_2 \ \text{by}(4) \\ &= y_2 \Rightarrow \text{Hence the proof} \end{aligned}$$

Theorem 2.1
Cancellation lows

- If $a, b, c \in \mathbb{R}$ and $a + c = b + c$ then $a = b$.
- If $a, b \in \mathbb{R}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

Remark

Let $x, y \in \mathbb{R}$ be non-zero elements and $x^{-1}, y^{-1} \in \mathbb{R}$. Then consider

$$\begin{aligned}(x^{-1}y^{-1})(yx) &= x^{-1}(y^{-1}y)x \\ &= x^{-1} \cdot 1 \cdot x \\ &= x^{-1}x \\ &= 1\end{aligned}$$

$(x^{-1}y^{-1})$ is the inverse of (yx).

$$\Rightarrow (yx)^{-1} = x^{-1}y^{-1}$$

For more convenient, we can write

$$(xy)^{-1} = y^{-1}x^{-1}.$$

Proposition 2.3

Let $x \in \mathbb{R}$. Then $x \cdot 0 = 0$.

Proposition 2.3
Proof

$$
\begin{aligned}
0 &= 0 + 0 \\
x \cdot 0 &= x \cdot (0 + 0) \\
&= x \cdot 0 + x \cdot 0
\end{aligned}
$$

By adding $-(x \cdot 0)$ to both side we get

$$
\begin{aligned}
-(x \cdot 0) + (x \cdot 0) &= [-(x \cdot 0) + (x \cdot 0)] + (x \cdot 0) \\
0 &= 0 + (x \cdot 0) \\
x \cdot 0 &= 0
\end{aligned}
$$

Proposition 2.4

Let $x, y \in \mathbb{R}$. Then $xy = 0 \Leftrightarrow x = 0 \text{ or } y = 0$ or equivalently $xy \neq 0 \Leftrightarrow x \neq 0 \text{ and } y \neq 0$.

If either $x = 0$ or $y = 0$ then $xy = 0$.

Conversely suppose that $xy = 0$ and $x \neq 0$.

Then x has x^{-1} in $\mathbb{R}$ such that

$$\begin{aligned} 0 &= x^{-1} \cdot 0 \\ &= x^{-1} \cdot (xy) \\ &= (x^{-1}x)y \\ &= 1 \cdot y \\ &= y \end{aligned}$$

Conversely suppose that $xy = 0$ and $y \neq 0$.

Then y has y^{-1} in $\mathbb{R}$ such that

$$\begin{aligned} 0 &= 0 \cdot y^{-1} \\ &= (xy) \cdot y^{-1} \\ &= x(yy^{-1}) \\ &= x \cdot 1 \\ &= x \end{aligned}$$

which completes the proof.

Proposition 2.5

Let $x, y, z \in \mathbb{R}$. Then, if $y \neq 0$ and $z \neq 0$

1. $\dfrac{x}{y} = \dfrac{xz}{yz}$
2. $\dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}$

Proposition 2.5

Proof

$$
\begin{aligned}
\frac{xz}{yz} &= (xz)(yz)^{-1} \\
&= xz\dot{z}^{-1}y^{-1} \\
&= x(zz^{-1})y^{-1} \\
&= xy^{-1} \\
&= \frac{x}{y}
\end{aligned}
$$

Proposition 2.5

Proof⇒Cont...

$$
\begin{aligned}
\frac{x+y}{z} &= (x+y)z^{-1} \\
&= z^{-1}(x+y) \\
&= z^{-1}x + z^{-1}y \\
&= xz^{-1} + yz^{-1} \\
&= \frac{x}{z} + \frac{y}{z}.
\end{aligned}
$$

Theorem 2.2

If $a, b, c, d \in \mathbb{R}$ and $c \neq 0,\ d \neq 0$.

1. $-(-a) = a$
2. $(c^{-1})^{-1} = c$
3. $(-1)a = -a$
4. $a(-b) = -(ab) = (-a)b$
5. $(-a) + (-b) = -(a+b)$
6. $\left(\frac{a}{c}\right)\left(\frac{b}{a}\right) = \frac{(ab)}{(cd)}$
7. $\frac{a}{c} + \frac{b}{d} = \frac{ad+bc}{cd}$

Poof is an exercise.

Order Axioms

What are order axioms?

- Apart from the algebraic properties mentioned earlier, the real numbers satisfy another important property called **order axioms**.
- Given any $x, y \in \mathbb{R}$ we can say whether $x = y$, $x < y$ or $x > y$.
- If $x > 0$ we say that x is positive.
- Less than/greater than relation between two numbers is known as oder relation on $\mathbb{R}$.

Order axiom

There exists a subset $\mathbb{P}$ of $\mathbb{R}$ called the set positive real numbers which satisfies the following properties.

(i) $0 \notin \mathbb{P}$.

(ii) For any $x \neq 0$, $x \in \mathbb{R}$, either $x \in \mathbb{P}$ or $-x \in \mathbb{P}$ but not both.

(iii) If $x, y \in \mathbb{P}$ then $x + y \in \mathbb{P}$ and $xy \in \mathbb{P}$.

Proposition 2.6

1 is a positive real number.

Proposition 2.6
Proof

$1 \neq 0$.

Therefore $-1 \in \mathbb{P}$ or $1 \in \mathbb{P}$.

If (-1) in $\mathbb{P}$, then $(-1)(-1) \in \mathbb{P} \Rightarrow 1 \in \mathbb{P}$.

But -1 and 1 both should not be in $\mathbb{P}$.

Therefore $-1 \notin \mathbb{P}$.

It implies that $1 \in \mathbb{P}$.

- The law of Trichotomy states that every real number is either positive, negative, or zero.
- That is if $x, y \in \mathbb{R}$ then exactly one of the following holds:

  (i) $x < y$ or
  (ii) $x = y$ or
  (iii) $x > y$.

Order properties of $\mathbb{R}$
Transitive property

If $x, y, z \in \mathbb{R}$ then

$$x < y \text{ and } y < z \Rightarrow x < z.$$

If $x \in \mathbb{R}$ then there exists a positive integer n such that $n > x$.

- The density property tells us that we can always find another real number that lies between any two real numbers.
- For example, between 7.61 and 7.62, there is 7.611, 7.612, 7.613 and so forth.
- That is if $a, b \in \mathbb{R}$ and $a < b$, then there exists $x \in \mathbb{R}$ such that $a < x < b$.

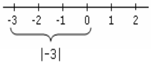
Remark

1. $a < b \wedge c > 0 \Rightarrow ac < bc$
2. $a < b \wedge c < 0 \Rightarrow ac > bc$
3. $0 < 1 \Rightarrow -1 < 0$
4. $a > 0 \Rightarrow \frac{1}{a} > 0$
5. $0 < a < b \Rightarrow 0 < \frac{1}{b} < \frac{1}{a}$
6. $a, b \in \mathbb{R}, p > 0 \Rightarrow a < b \Leftrightarrow a^p < b^p$.

- The absolute value of any $x \in \mathbb{R}$ is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

- The absolute value of a number may be thought of as its distance from zero.



- For any $x \in \mathbb{R}$, $|x|$ is always non-negative and we have:
  1. $|x| \geq x$
  2. $|x| \geq -x$
- Further, $|x| = |-x|$ and
- $|x^n| = |x|^n$.

For any $x, y \in \mathbb{R}$, the following properties hold.

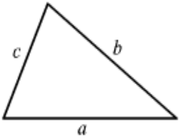
1. $|x| = |-x| \geq 0$. The equality holds if and only if $x = 0$.
2. $|x|^2 = x^2 = (-x)^2$.
3. $|x \cdot y| = |x| \cdot |y|$.

Triangle inequality

A property for any triangle

- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- In the figure, the following inequalities hold.

  1. $a + b > c$
  2. $a + c > b$
  3. $b + c > a$



Triangle inequality
Motivative example 1

Check whether it is possible to have a triangle with the given side lengths 4, 5, 7.

We should add any two sides and see if it is greater than the other side.

- The sum of 4 and 5 is 9 and 9 is greater than 7.
- The sum of 4 and 7 is 11 and 11 is greater than 5.
- The sum of 5 and 7 is 12 and 12 is greater than 4.
- These sides 4, 5, 7 satisfy the above property.
- Therefore, it is possible to have a triangle with sides 4, 5, 7.

Triangle inequality
Motivative example 2

Check whether the given side lengths form a triangle 2, 5, 9.

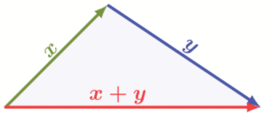
Check whether the sides satisfy the above property.

- The sum of 2 and 5 is 7 and 7 is less than 9.
- This set of side lengths does not satisfy the above property.
- Therefore, these lengths do not form a triangle.

Triangle inequality
Theorem 2.3

- **The triangle inequality states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side**.
- The triangle inequality requires that the absolute value satisfy for any real numbers x and y:

$$|x + y| \leq |x| + |y|.$$



Triangle inequality

Theorem 2.3 ⇒ Proof

We must show that for any $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

$$
\begin{aligned}
|x+y| &= (x+y) \text{ or } -(x+y) \\
\text{If } |x+y| &= x+y \\
&\leq |x| + |y| \quad (5) \\
\text{If } |x+y| &= -(x+y) \\
&= -x-y \\
&= (-x)+(-y) \\
&\leq |x| + |y| \quad (6)
\end{aligned}
$$

From (5) and (6) we have

$$|x+y| \leq |x| + |y|.$$

For any $a_1, a_2, ..., a_n \in \mathbb{R}$

$$\begin{aligned} |a_1 + a_2 + ... + a_n| &\leq |a_1| + |a_2| + ... + |a_n| \\ \text{i.e} \left|\sum_{i=1}^{n} a_i\right| &\leq \sum_{i=1}^{n} |a_i| \end{aligned}$$

Triangle inequality

Example 1

Show that

$$|a| \leq 1 \Rightarrow |3 + a^3| \leq 4.$$

$$
\begin{aligned}
|3+a^3| &\le |3|+|a^3| \\
&= 3+|a|^3 \\
&\le 3+1^3 \ (\text{since } |a|\le 1) \\
&= 4.
\end{aligned}
$$

Triangle inequality
Example 2

Use the triangle inequality to prove that:

$$|a| \leq \frac{1}{2} \Rightarrow |a+1| \leq \frac{3}{2}.$$

Triangle inequality

Example 2 ⇒ Solution

$$
\begin{aligned}
|a+1| &\leq |a|+|1| \\
&= |a|+1 \ (\text{since } |a| \leq \frac{1}{2}) \\
&\leq \frac{1}{2}+1 \\
&= \frac{3}{2}.
\end{aligned}
$$

The reverse form of triangle inequality

The reverse triangle inequality which states that for any real numbers x and y:

$$|x - y| \geq \Big| |x| - |y| \Big|.$$

The reverse form of triangle inequality

Proof

$$
\begin{aligned}
\Big||x|-|y|\Big| &= |x|-|y| \text{ or } -(|x|-|y|) \\
&= |x|-|y| \text{ or } |y|-|x| \\
|x| &= |x-y+y| \\
|x| &\le |x-y|+|y| \\
|x|-|y| &\le |x-y| \qquad (7) \\
|y| &= |y-x+x| \\
|y| &\le |y-x|+|x| \\
|y|-|x| &\le |y-x| \\
|y|-|x| &\le |x-y| \qquad (8)
\end{aligned}
$$

From (7) and (8) we have

$$|x-y| \ge \Big||x|-|y|\Big|.$$

The reverse form of triangle inequality

Example 1

Show that

$$|b| < 1 \Rightarrow |3 - b| > 2.$$

The reverse form of the triangle inequality then gives

$$\begin{aligned} |3-b| &\geq \Big||3|-|b|\Big| \\ &= \Big|3-|b|\Big| \\ &\geq 3-|b|. \end{aligned}$$

Now $|b| < 1$, so that $-|b| > -1$. Thus

$$\begin{aligned} 3-|b| &> 3-1 \\ &= 2. \end{aligned}$$

The reverse form of triangle inequality
Example 2

Use the reverse form of the triangle inequality to prove that:

$$|b| < \frac{1}{2} \Rightarrow |b^3 - 1| > \frac{7}{8}.$$

$$\begin{aligned} |b^3 - 1| &\geq \left| |b^3| - |1| \right| \\ &= \left| |b^3| - 1 \right| \\ &\geq -(|b^3| - 1) \\ &= -(|b|^3 - 1) \\ &= 1 - |b|^3 \end{aligned}$$

Now $|b| < \frac{1}{2}$, so that $|b|^3 < \frac{1}{8}$ and $-|b|^3 > -\frac{1}{8}$. Thus

$$|b^3 - 1| > 1 - \frac{1}{8} = \frac{7}{8}.$$

Inequalities involving n

- In Analysis we often need to prove inequalities involving an integer n.
- It is common convention in mathematics that the symbol n is used to denote an integer (frequently a natural number).

Prove that

$$2n^2 \geq (n+1)^2, \text{ for } n \geq 3.$$

n	1	2	3	4
$2n^2$	2	8	18	32
$(n+1)^2$	4	9	16	25

Rearranging this inequality into an equivalent form, we obtain

$$\begin{aligned} 2n^2 \geq (n+1)^2 &\Leftrightarrow 2n^2 - (n+1)^2 \geq 0 \\ &\Leftrightarrow n^2 - 2n - 1 \geq 0 \\ &\Leftrightarrow (n-1)^2 - 2 \geq 0 \\ &\Leftrightarrow (n-1)^2 \geq 2. \end{aligned}$$

This final inequality is clearly true for $n \geq 3$, and so the original inequality $2n^2 \geq (n+1)^2$ is true for $n \geq 3$.

Induction Principle

Why do we need mathematical induction?

- Mathematical induction is a method of mathematical proof typically used to establish a given statement for all natural numbers.
- We shall discuss more inequalities after studying the induction principle.
- Let us start with a discussion of bounded sets in $\mathbb{R}$.

Upper bound and lower bound
Definition

Let S be a nonempty set of real numbers, that is $\emptyset \neq S \subset \mathbb{R}$. If there is $m \in \mathbb{R}$ such that $m \leq x$ for every $x \in S$, we say m is **lower bound** for the set S. Similarly, a real number M is said to be an **upper bound** for S if $x \leq M$, for every $x \in S$.

Upper bound and lower bound
Example

Determine an upper bound and a lower bound of the set $S = \{2, 5, 8, 13\}$.

Upper bound and lower bound

Example $\Rightarrow$ Solution

- The given set is $S = \{2, 5, 8, 13\}$.
- The number 13 is an upper bound of S.
- The number 2 is a lower bound of S.

- The set $\mathbb{R}$ of all real numbers has neither an upper bound or a lower bound.
- The reason is that for any given real M, there is a number in the set $\mathbb{R}$, greater than or equal to M; for example, $M + 1$.
- Thus no number M can be an upper bound for $\mathbb{R}$.
- Similar reasoning shows that $\mathbb{R}$ has no lower bound either.

The set $\mathbb{Z}$ of integers also has neither an upper bound nor a lower bound.

- If a set has an upper bound, it will have several upper bounds.
- For example, if M is an upper bound for S, then so is the number $M + 1$ or the number $M + 100$ and so on.
- The same is true with lower bounds.

- Consider the set $S = \{2, 5, 8, 13\}$.
- The number 13 is an upper bound. Any number ≥ 13 is also an upper bound. The least upper bound is 13.
- The number 2 is a lower bound. Any number ≤ 2 is also a lower bound. The greatest lower bound is 2.

Upper bound and lower bound
Remark 4

- A set may have a lower bound but not any upper bound and vice-versa.
- For example, the set $\mathbb{N}$ has zero as a lower bound but has no upper bound.

A nonempty set $S \subset \mathbb{R}$, is said to be **bounded above** if there is a real M such that $x \leq M$, for every $x \in S$. It is said to be **bounded below** if there is a real number m such that $m \leq x$, for every $x \in S$. If a set is bounded above and below, it is called a **bounded set**. The empty set $\emptyset$ is always taken as a bounded set.

- A finite set is always bounded.
- For example the set $S = \{2, 5, 8, 13\}$ is bounded.
- The number 13 is an upper bound. So, the set is bounded above.
- The number 2 is a lower bound. So, the set is bounded below.
- Since it is both bounded above and bounded below, we call it as a bounded set.

Bounded sets
Example 1

Show that the set $\{x^2 + 1 : x \in \mathbb{R}\}$ is bounded below.

For any real $x, x^2 \geq 0$ so $x^2 + 1 \geq 1$.

It implies that 1 is a lower bound for the set.

Therefore the set is bounded below.

Bounded sets
Example 2

Show that the set $\{\dfrac{1}{x^2+1} : x \in \mathbb{R}\}$ is bounded.

Bounded sets

Example 2 ⇒ Solution

For any real $x, x^2 \geq 0$ so $x^2 + 1 \geq 1$.

Taking reciprocals, we have $\dfrac{1}{x^2+1} \leq 1$.

Since $x^2 + 1 > 0$, we have, for any real x,

$$0 < \frac{1}{x^2+1} \leq 1.$$

In other words, 0 and 1 are the lower and the upper bounds for the set $\{\dfrac{1}{x^2+1} : x \in \mathbb{R}\}$.

Therefore it is bounded.

Show that the set of numbers of the form

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n},$$

where $n \geq 1$ is an integer, form a bounded set.

Bounded sets

Example 3 ⇒ Solution

If we let

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n}, \text{ then} \\ \frac{1}{2}S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} ... + \frac{1}{2^{n+1}}, \end{aligned}$$

so

$$S_n - \frac{1}{2}S_n = (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n}) - (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} ... + \frac{1}{2^{n+1}})$$

hence

$$\begin{aligned} \frac{1}{2}S_n &= 1 - \frac{1}{2^{n+1}} \\ S_n &= 2\left(1 - \frac{1}{2^{n+1}}\right) \leq 2. \end{aligned}$$

Therefore $M = 2$ is an upper bound on the given set, while all the numbers are positive, so 0 is a lower bound.

Bounded sets
Remark 2

- If a set is bounded below, its lower bound may or may not be in that set.
- Thus, if $S = \{x : x \geq 0\}$ is the set of non-negative numbers, 0 is a lower bound to the set and it belongs to the set.
- But -1 is also a lower bound to the set not belonging to the set.
- Similarly, an upper bound of a set may or may not be in the set S.

- We say m is the **least element** of a set S if it is a lower bound of S and also is in the set S.
- Note that if a set has a least element, it must be unique.
- The **greatest element** of a set is analogously defined.
- It is also unique, when it exists.

A set may be bounded above, without having the greatest element or may be bounded below, without having the least element.

- The set $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is bounded but has no greatest or the least element in the set itself.
- 0 is a lower bound but it is not in the set where as 1 is an upper bound which is not in the set.
- Hence $(0,1)$ is a bounded set that does not have greatest and least element.

- The set $\mathbb{N} = \{1, 2, ..., n, ...\}$ of natural numbers is bounded below and infact has the least element, namely 1. Every non-empty subset of $\mathbb{N}$ also has this property.
- That is every non-empty subset of $\mathbb{N}$ has the least element. It is called **well-ordering axiom** of natural numbers.

Infimum of a set
Definition

Let $S \subset \mathbb{R}$ be non-empty. If there exists $m \in \mathbb{R}$, such that

1. $m \leq x$ for every $x \in S$, and
2. if m_1 is any lower bound of S, then $m_1 \leq m$,

then m is called **infimum** of S and we write $m = \inf S$. It is also called **greatest lower bound** of S and we write,
$m = \inf S = \text{g.l.b. } S$.

Let $S \subset \mathbb{R}$ be non-empty. If there exists $M \in \mathbb{R}$, such that

1. $x \leq M$ for every $x \in S$, and
2. if M_1 is any upper bound of S, then $M \leq M_1$,

then M is called **supremum** of S and we write $M = \sup S$. It is also called **least upper bound** of S and we write,
$M = \sup S = \text{l.u.b. } S$.

Infimum and supremum

Remark 1

The infimum of a set is unique when it exists.

- Suppose both m and m_1 are infima of the set S.
- Then both are lower bounds of S.
- Since m is the greatest of the lower bounds, $m_1 \leq m$.
- By the same argument, $m \leq m_1$.
- In other words, $m = m_1$.

Like the infimum of a set, the supremum of a set, when it exists, is unique.

Infimum and supremum

Remark 2

- The words **lower bound**, **least element**, **infimum** all have precise and distinct meanings.
- Analogous statements are also true regarding the **supremum** of a set.

Infimum and supremum
Remark 3

- If a set has the least element, it is also its infimum but not conversely.
- If a set has greatest element, it is its supremum but not conversely.

Infimum and supremum
Example 1

Consider the set $S = (-1, 5)$ and determine followings:

1. Upper bounds
2. Greatest element
3. Supremum

1. The set $S = (-1, 5)$ is bounded above by 100, 35, 6, 5.55, 5. In fact 5 is its least upper bound.
2. 5 is an upper bound of S which is not in the set S. Hence $S = (-1, 5)$ does not have a greatest element.
3. The set $S = (-1, 5)$ has a supremum and it is equal to 5.

Infimum and supremum
Example 2

Consider the set $S = (-1, 5]$ and determine followings:

1. Upper bounds
2. Greatest element
3. Supremum

1. The set $S = (-1, 5]$ is bounded above by 100, 35, 6, 5.55, 5. In fact 5 is its least upper bound.
2. 5 is an upper bound of S which is in the set S. Hence $S = (-1, 5]$ does have a greatest element and it is equal to 5.
3. The set $S = (-1, 5]$ has a supremum and it is equal to 5.

Find the infimum and supremum of following sets:

1. If $a < b$, then $S = [a, b]$
2. If $a < b$, then $S = [a, b)$
3. If $a < b$, then $S = (a, b]$
4. If $a < b$, then $S = (a, b)$

Infimum and supremum

Example 3 ⇒ Solution

1. inf $S=a$ and sup $S=b$
2. inf $S=a$ and sup $S=b$
3. inf $S=a$ and sup $S=b$
4. inf $S=a$ and sup $S=b$

Find the infimum and supremum of following sets:

1. $S = \{1, 2, 3, 4, 5, 6\}$
2. $S = \{q \in \mathbb{Q} : e < q < p\}$
3. $S = \{x \in \mathbb{R} | 0 < x \leq 1\}$
4. $S = \{1/n | n \in \mathbb{N}\}$

Infimum and supremum

Example 4 ⇒ Solution

1. inf S=1 and sup S=6
2. inf S=e and sup S=p
3. inf S=0 and sup S=1
4. inf S=0 and sup S=1

Infimum and supremum
Remark 4

- If a set S is not bounded below, clearly it will not have the infimum.
- We then say $\inf S$ does not exists.
- Similarly, if it is not bounded above, $\sup S$ does not exists.
- But even if a set S is bounded below, it is not immediately clear that $\inf S$ exists, or if it is bounded above, that $\sup S$ exists.
- But below axiom assumes thier existence.

Let S be a non-empty subset of $\mathbb{R}$.

- If S is bounded below, then $\inf S$ exists in $\mathbb{R}$.
- If S is bounded above, then $\sup S$ exists in $\mathbb{R}$.

Theorem 2.4

A real number m is the infimum of a non-empty set S if and only if both the conditions below are satisfied.

1. $m \leq x$, for every $x \in S$
2. for any $\epsilon > 0$, there is $x \in S$, such that $m \leq x < m + \epsilon$.

Theorem 2.4
Proof

Suppose m=inf S.

Then certainly $m \leq x$ for all $x \in S$.

So (1) is satisfied.

Let any $\epsilon > 0$ be given.

If $m + \epsilon \leq x$ for every $x \in S$, then $m + \epsilon$ will be a lower bound for S, so m cannot be the greatest lower bound for S.

Hence there is at least some $x \in S$ such that $x < m + \epsilon$.

This shows that (2) is also satisfied when m=inf S.

In other words, condition (1) and (2) are necessary.

Theorem 2.4

Proof ⇒ Cont...

Now, let us show that they are also sufficient.

Suppose there is a real m which satisfies both (1) and (2) above.

Then by (1), m is a lower bound of S.

Let $m_1 > m$ be arbitary and let $\epsilon = m_1 - m$.

Then $\epsilon > 0$ and by (2), for $m_1 = m + \epsilon$, there is $x \in S$ such that $m \leq x < m + \epsilon = m_1$.

This shows that m_1 cannot be a lower bound of S, that is, no number $> m$ can be a lower bound for S, so that m=inf S.

This shows that the given conditions are also suffucient.

Theorem 2.5

A real number M is the supremun of non-empty set S if and only if both the conditions below are satisfied.

1. $x \leq M$, for every $x \in S$
2. for any $\epsilon > 0$, there is $x \in S$, such that $M - \epsilon < x \leq M$.

Example 1

Let $S = (0, \infty)$. Then prove that $\inf S = 0$.

Since $(0, \infty)$ consists of all real numbers greater than 0, 0 is a lower bound of $(0, \infty)$.

Let $\epsilon > 0$.

Then $\dfrac{\epsilon}{2} \in (0, \infty)$ and

$$\frac{\epsilon}{2} < 0 + \epsilon.$$

Hence 0 satisifes the alternate characterization of infimum given in above Theorem 2.4, so $\inf\,(0, \infty) = 0$

Example 2

Let $S = [1, 2]$. Then prove that sup S=2 and inf S=1.

let $\epsilon > 0$ be given.

$S = \{x|1 \leq x \leq 2\}$.

Clearly $x \leq 2 \ \forall x \in S \Rightarrow 2$ is an upper bound.

We consider $2 - \epsilon$ and the inequality.

$2 - \epsilon < x_0 \Rightarrow$ Now we define x_0 as $x_0 = 2 - \frac{\epsilon}{2}$.

Then clearly $2 - \frac{\epsilon}{2} = x_0 \in S$.

Therefore 2 is the supremum.

In the same way you can show that $\inf S{=}1$.

Suppose $P(n)$ is a statement about a positive integer n and suppose that

1. $P(1)$ is true and
2. if $P(n)$ is true, so is $P(n+1)$.

Then the statement $P(n)$ is true for all $n \in \mathbb{N}$.

Let S denote the set of natural numbers for which given statement $P(n)$ is false.

Thus, $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$.

So S is a subset of $\mathbb{N}$.

The theorem will be proved if we show that S has no elements in it, that is, $S = \emptyset$.

Suppose that $S \neq \emptyset$, so it is non-empty subset of $\mathbb{N}$ and so by well-ordering axiom, it has least element, say k.

Now $k \neq 1$, since by Hypothesis (1), the statement $P(1)$ is true.

Hence $k > 1$ and hence $k - 1$ is a positive integer.

Now $P(k - 1)$ is true, for otherwise, $k - 1$ would be in S and k will not be the least element of S.

But if $P(k-1)$ is true, so is $P(k) = P((k-1)+1)$, by (2) of the Induction Hypothesis.

But then $k \notin S$, contradicting the fact that $k \in S$.

This contradiction shows that S must be empty; that is, the statement $P(n)$ must be true for all $n \in \mathbb{N}$.

Principle of mathematical induction
Example 1

For each integer $n \geq 1$

1. $4^n + 5$ is divisible by 3,
2. $4^n + 15n - 1$ is divisible by 9.

The statement holds when $n = 1$ because then $4^n + 5 = 4^1 + 5 = 9$.

Suppose the statement is true for n, say $4^n + 5 = 3k$ for some integer k.

Then

$$\begin{aligned} 4^{n+1} + 5 &= 4^n \cdot 4 + (20 - 15) \\ &= 4(4^n + 5) - 15 \\ &= 12k - 15 \\ &= 3(4k - 5), \end{aligned}$$

a multiple of 3.

This shows that the result is also holds for $n + 1$.

Hence, by the induction theorem, the statement holds for all $n \geq 1$.

Again, it is clear that the result holds when $n = 1$.

Suppose it holds for n.

To see it also holds for $n+1$, note that

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= (3+1) \cdot 4^n + 15n + 15 - 1 \\ &= (4^n + 15n - 1) + 3(4^n + 5). \end{aligned}$$

By induction hypothesis, the first term is divisible by 9 while by part (i), $(4^n + 5)$ is divisible by 3.

So $3(4^n + 5)$ is divisible by 9.

Hence, the left side is also so.

Use induction to establish the formula

$$1 + 2 + \dots + n = \frac{n \cdot (n+1)}{2} \text{ for all } n \in \mathbb{N}.$$

To show that the formula holds when $n = 1$, we have to check that

$$1 = \frac{1 \cdot (1+1)}{2} \Leftrightarrow 2 = 1 + 1,$$

and this is certainly true.

In particular, the given formula does hold when $n = 1$.

Suppose now that the formula holds for some $n \in \mathbb{N}$, namely suppose that

$$1 + 2 + ... + n = \frac{n \cdot (n+1)}{2}.$$

Adding $n+1$ to both sides and simplifying, we then get

$$\begin{aligned} 1 + 2 + ... + (n+1) &= \frac{n \cdot (n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

This shows that the formula holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$.

Prove that

$$2^n \geq n^2 \quad \text{for } n \geq 4.$$

Let $P(n)$ be the statement $P(n) : 2^n \geq n^2$.

$$P(4) \Rightarrow \quad \begin{aligned} 2^4 &= 16 \\ 4^2 &= 16 \end{aligned}$$

It implies that $2^n \geq n^2$ holds for $n = 4$.

Suppose the statement is true for n.

$$2^n \geq n^2$$

Multiplying above by 2 we get

$$2^{n+1} \geq 2n^2$$

Hence it is enough to show that $2n^2 \geq (n+1)^2$.

$$\begin{aligned} 2n^2 \geq (n+1)^2 &\Leftrightarrow 2n^2 \geq n^2 + 2n + 1 \\ &\Leftrightarrow n^2 - 2n - 1 \geq 0 \\ &\Leftrightarrow (n-1)^2 - 2 \geq 0 \end{aligned}$$

This is true for $n \geq 4$.

Therefore $2^{n+1} \geq (n+1)^2$.

So the statement is true for $(n+1)$.

It follows from mathematical induction that the statement $P(n)$ is true for $\forall n \in \mathbb{N}, n \geq 4$.

Thank you !