Calculus (Real Analysis I)

(MAT122β)

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

General information about the course unit

- Course unit: Calculus (Real Analysis I)(MAT122β)
- Lecturer: A.W.L. Pubudu Thilan
- Tutorial Instructor: Manori Priyadarshani
-

Tuesday	10.00-12.00	Lecture
Wednesday	08.00-09.00	Tutorial class

General information about the course unit
Cont...

- Credit value: 2.5
- Number of lecture hours: 30
- Number of tutorial hours: 15
- Attendance: Both tutorial and lecture will be considered
- Method of assessment: Two hour written examination at the end of semester.

References

1. David Branan, A first course in Mathematical Analysis, Cambridge University Press, 2006.
2. Deshpande, J.V., Mathematical Analysis and Applications (An Introduction), Narosha Publishing House, India 2005.
3. Any elementary book on Mathematical Analysis and/or Calculus.
4. http://www.math.ruh.ac.lk/∼pubudu/

Course content

1. **Chapter 1** Elementary Logic
2. **Chapter 2** The Real Number System
3. **Chapter 3** Sequences
4. **Chapter 4** Limit and Continuity of Functions
5. **Chapter 5** Differentiability

Elementary Logic

Why do we need logic?

- Mathematics is a language.
- It is used to describe the world around us.
- In order to understand mathematics, like any other language we must learn the vocabulary and how to express ideas with that vocabulary.
- Mathematical logic is the structure that allows us to describe concepts in terms of mathematics.

Why do we need logic?
Cont...

- In this Chapter we will discuss some notions of (symbolic) logic and thier importance of mathematical proofs and some topics on set theory.
- All mathematical results are obtained by logical deductions from previously obtained results which are now accepted as true, or from the axioms which have been assumed to be true.
- Therefore, the logic plays an important role in Mathematical analysis.

Propositions

What is proposition?

- **A proposition is a declarative statement (or expression) which is either true or false (but not both)**.
- For example, "Today is Tuesday" is a proposition.
- This statement can be true or false, but not both.

Notations

- It is common to define a shorthand notation for propositions.
- For example, the letter P can be used to denote the proposition "Today is Monday".
- If the statement is true, then P has truth value true and it is denoted by "True" or "T".
- If it is false, then P has truth value false and it is denoted by "False" or "F".

Examples

Determine which of the following are propositions. If a given statement is proposition, then write down its truth value.

(i) $P : 2+3=5$

(ii) $Q : 2+2=3$

(iii) $R : x+1=2$

(iv) S : "π is rational"

(v) T : "Read this lecture notes carefully"

(i) $P : 2 + 3 = 5 \Rightarrow P$ is a proposition and it takes truth value "True".

(ii) $Q : 2 + 2 = 3 \Rightarrow Q$ is a proposition and it takes truth value "False".

(iii) $R : x + 1 = 2 \Rightarrow R$ is not a proposition because it is neither true or false since the variable x has not been assigned values.

(iv) S : "π is rational" $\Rightarrow S$ is a proposition and it takes truth value "False".

(v) T : "Read this lecture notes carefully" $\Rightarrow T$ is not a proposition because it is not a declarative sentence.

How can we use logic?

- Merely it is not the task of logic to decide whether a given statement is true or false.
- In additon to that we can use logic to obtain very important mathematical results.
- Next we shall discuss some ways to connect propositions to form new propositions.
- It will help us in obtaining different mathematical results as logical decuctions.

Logical symbols

The first logical symbols we use in this course unit are listed in the following table.

Symbol	**Read as**
$\sim$	NOT
$\wedge$	AND
$\vee$	OR
$\rightarrow$	IMPLIES
$\leftrightarrow$	IF AND ONLY IF

Logical symbols
Cont...

- The above symbols are used to build new statements (propositions) using old statements.
- Thus these symbols are called **logical operators**.
- Moreover, the operator $\sim$ is known as **unary operator** since it operates on one statement.
- The other four operators are ofetn called **binary operator** or **connectives**, connecting two statements P and Q.

Definition 1.1
NOT

If P is a statement then the negation of P is denoted by $\sim P$ and is read as *not* P. This means that if P is true then $\sim P$ is not true and if P is false then $\sim P$ is true.

This result can be represented using the following truth table.

P	$\sim P$
T	F
F	T

Moreover, the statement *not not* P, i.e. $\sim(\sim P)$ means the same as P.

Example 1.1

1. **Example 1**

P:	The number 3 is odd	True
$\sim P$:	The number 3 is not odd	False

2. **Example 2**

Q:	$5 < 3$	False
$\sim Q$:	$5 \geq 3$	True

Definition 1.2
CONJUNCTION

We say that the proposition $P \wedge Q$ (or P and Q) is true, if the two propositions P and Q both are true, and is false otherwise.

The corresponding truth table is as follows:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.2

Determine truth values of following statements.

1. "Red is a color and $2 + 4 = 6$."
2. "One day = exactly 23 hours and one hour = exactly 60 minutes."
3. "$5 + 6 = 4$ and $6-3=5$."

Example 1.2

Solution

1. "Red is a color and $2 + 4 = 6$." (T and T = T)
   Since both facts are true, the entire sentence is true.

2. "One day = exactly 23 hours and one hour = exactly 60 minutes." (F and T = F)
   Since the first fact is false, the entire sentence is false.

3. "$5 + 6 = 4$ and 6-3=5." (F and F = F)
   Since both facts are false, the entire sentence is false.

Definition 1.3
DISJUNCTION

We say that the proposition $P \vee Q$ (or P or Q) is true, if at least one of two propositions P and Q is true, and is false otherwise.

The corresponding truth table is as follows:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1.3

Determine truth values of following statements.

1. "Red is a color or $2 + 4 = 6$."
2. "One day $=$ exactly 23 hours or one hour $=$ exactly 60 minutes."
3. "$5 + 6 = 4$ or $6\text{-}3\text{=}5$."

Example 1.3
Solution

1. "Red is a color or $2 + 4 = 6$." (T or T = T)
   Since both facts are true, the entire sentence is true.

2. "One day = exactly 23 hours or one hour = exactly 60 minutes." (F or T = T)
   Since the second fact is true, the entire sentence is true.

3. "$5 + 6 = 4$ or $6-3=5$." (F or F = F)
   Since both facts are false, the entire sentence is false.

Definition 1.4
CONDITIONAL

We say that proposition $P \to Q$ (or if P then Q, or P implies Q) is true, if the sentence P is false or if the sentence Q is true or true both, and is false otherwise.

It should be noted that the proposition $P \rightarrow Q$ is false only if P is true and Q is false. To understand this, note that if we draw a false conclusion from a true assumption, then our argument must be faulty. On the other hand, if our assumption is false and the conclusion is true, the argument may still be acceptable.

Below is the coressponding truth table.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.4

Determine truth values of following statements for $P \to Q$.

1. $P : 3 + 3 = 3$ and $Q : 2 + 5 = 3$.
2. $P : 3 + 3 = 6$ and $Q : 2 + 5 = 3$.
3. $P : 3 + 3 \neq 6$ and $Q : 2 + 5 = 3$.
4. $P : 5 > 3$ and $Q : 5 > 2$.
5. $P : 1.5 > 2$ and $Q : 1.5 > 1$.
6. $P : 0.5 > 2$ and $Q : 0.5 > 1$.

Example 1.4
Solution

1. $P : 3 + 3 = 3$ and $Q : 2 + 5 = 3$ then $P \to Q$ is True.
2. $P : 3 + 3 = 6$ and $Q : 2 + 5 = 3$ then $P \to Q$ is False.
3. $P : 3 + 3 \neq 6$ and $Q : 2 + 5 = 3$ then $P \to Q$ is True.
4. $P : 5 > 3$ and $Q : 5 > 2$ then $P \to Q$ is True.
5. $P : 1.5 > 2$ and $Q : 1.5 > 1$ then $P \to Q$ is True.
6. $P : 0.5 > 2$ and $Q : 0.5 > 1$ then $P \to Q$ is True.

Definition 1.5
BICONDITIONAL

We say that the proposition $P \leftrightarrow Q$ (read as P if and only if Q) is true, if the two propositions P, Q both are true or both are false, and is false otherwise.

The corresponding truth table is as follows:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

It is remarked here that the proposition $P \leftrightarrow Q$ is the same as the proposition $(P \to Q) \wedge (Q \to P)$.

Example 1.5

Determine truth values of following statements for $P \leftrightarrow Q$.

1. $P: 2+2=4$ and $Q: \pi$ is irrational
2. $P: 2+2=4$ and $Q: 1+3=5$
3. $P: 2+2\neq 4$ and $Q: 1+3=5$

Example 1.5
Solution

1. $P : 2 + 2 = 4$ and $Q : \pi$ is irrational then $P \leftrightarrow Q$ is True.
2. $P : 2 + 2 = 4$ and $Q : 1 + 3 = 5$ then $P \leftrightarrow Q$ is False.
3. $P : 2 + 2 \neq 4$ and $Q : 1 + 3 = 5$ then $P \leftrightarrow Q$ is True.

Summary

The above five defintions can be summarized in the following table.

P	Q	$P \wedge Q$	$P \vee Q$	$\sim P$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	T	T	F	T	T
T	F	F	T	F	F	F
F	T	F	T	T	T	F
F	F	F	F	T	T	T

Truth table for complicated compound statements

- Now we look at truth table of more complicated compound statements.
- To do so we first list the underline component statements P, Q, R, and so on in lexicographical order.
- We then proceed working inside out and step by step to construct the resulting truth values of the desired compound statement for each possible combination of the component statements.
- It is enough to use the actions of each operations summarized in the previous truth table.

Example 1.6

Construct a truth table for $P \to [\sim (Q \vee R)]$.

Here we need $2^3 = 8$ combinations of truth values for the underline component statements P, Q, and R. Once we list all those combinations in a truth table we compute $Q \vee R$, $\sim (Q \vee R)$ and the required statement $P \to [\sim (Q \vee R)]$.

P	Q	R	$Q \vee R$	$\sim (Q \vee R)$	$P \to [\sim (Q \vee R)]$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Example 1.6

Construct a truth table for $P \to [\sim (Q \vee R)]$.

P	Q	R	$Q \vee R$	$\sim (Q \vee R)$	$P \to [\sim (Q \vee R)]$
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	T	T

Tautologies and Contradictions

Definition 1.6

A compound proposition that is always true, no matter what the truth values of the propositions that occur in, is called a tautology (i.e. a tautology is a proposition which is always true). A compound proposition that is always false, no matter what the truth values of the propositions that occur in, is called a contradiction (i.e. a contradiction is a proposition which always false).

Example 1.6

1. $P \wedge \sim P$ is a contradiction.
2. $P \vee \sim P$ is a tautology.

Logically Equivalence

Definition 1.7

We say that a proposition P is logically equivalent to a proposition Q if and only if $P \leftrightarrow Q$ is a tautology. We denote it by $P \equiv Q$.

Remark

- For example, $P \to Q$ is logically equivalent to $\sim Q \to \sim P$. The latter is known as the contrapositive of the former.
- The proposition $P \to Q$ is not logically equivalent to $Q \to P$, where the latter one is known as the converse of the former.
- Moreover, the proposition $\sim P \to \sim Q$ is the inverse of $P \to Q$ and they are not equivalent to each other.

Precedence of Logical Operators

Precedence of logical perators

Operator	Precedence
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Examples:

1. $\sim P \wedge Q \equiv (\sim P) \wedge Q$
2. $P \wedge Q \vee R \equiv (P \wedge Q) \vee R$

Theorem 1.1
Distributive law

Let P, Q, R be propositions. Then the followings hold.

1. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
2. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Theorem 1.2
De Morgan's law

Let P, Q be propositions. Then the followings hold.

1. $\sim (P \wedge Q) \equiv (\sim P \vee \sim Q)$
2. $\sim (P \vee Q) \equiv (\sim P \wedge \sim Q)$

Theorem 1.3
Associative law

Let P, Q, R be propositions. Then the followings hold.

1. $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
2. $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

Exercise 1.1

Construct truth tables for the following statements:

1. $(\sim P) \leftrightarrow (\sim Q)$ (Compare to $P \leftrightarrow Q$)
2. $[P \vee (\sim Q)] \rightarrow P$
3. $\sim [P \wedge (Q \vee R)]$

Exercise 1.2

Decide which are tautologies, which are contradictions and which are neither. Try to decide using intuition and then check with truth tables.

1. $P \rightarrow P$
2. $P \leftrightarrow (\sim P)$
3. $P \rightarrow (\sim P)$
4. $(P \wedge \sim P) \rightarrow Q$
5. $(P \wedge Q) \rightarrow P$
6. $(P \vee Q) \rightarrow P$
7. $P \rightarrow (P \wedge Q)$
8. $P \rightarrow (P \vee Q)$

Propositional Functions

What is propositional functions?

In many instances we have propositions like **x is even** which contains one or more variables. We shall call them propositional functions. It is clear that the expression **x is even** is true for only certain values of x and is false for others. In this point various questions may arise:

1. What values of x do we permit?
2. Is the statement true for all such values of x in question?
3. Is the statement true for some such values of x in question?

What is propositional functions?

Cont...

To answer the first question we need the notion of a universe (domain of x). At this point we have to consider sets. An important thing about a set is what it contains (what are the elements of a set?) If P is a set and x is an element of P, then we write $x \in P$. A set is usually described by the following two ways.

1. by enumeration. e.g. $\{1, 2, 3\}$ denotes the set consisting of the elements 1,2, and 3.
2. by a defining property $p(x)$. $P = \{x | x \in U, p(x) \text{ is true}\}$ or simply $P = \{x | p(x)\}$.

Example 1.7

(a) $\mathbb{N} = \{1, 2, 3, ...\}$=set of natural numbers.

(b) $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$=set of integers.

(c) $\mathbb{P} = \{x | x \in \mathbb{Z}, -3 < x < 2\} = \{-2, -1, 0, 1\}$.

Mathematical Statements

What is mathematical statement?

In a mathematical proof or discussion one makes various assertions often called statements or sentences. For example

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $3x^2 + 2x - 1 = 0$
3. if $n(\geq 3)$ is an integer then $a^n + b^n = c^n$ has no positive integer solutions
4. the derivative of the function x^2 is $2x$.

What is mathematical statement?
Cont...

Although a mathematical statement always has a very precise meaning, certain things are often assumed from the context in which the statement is made. For example, the statement (1) may be an abbreviation for the statement

- for all real numbers x and y, $(x+y)^2 = x^2 + 2xy + y^2$,

or it may be an abbreviation of

- for all complex numbers x and y, $(x+y)^2 = x^2 + 2xy + y^2$.

Remark: The precise meaning should always be clear from the context, if it is not then more information should be provided.

Quantifier Logic

Definition 1.8

- **The expression <u>for all</u> (for every, or for each, or (sometimes) for any), is called the <u>universal quantifier</u> and is denoted by** $\forall$.
- **The expression <u>there exists</u> (or there is, or there is at least one, or there are some), is called <u>existential quantifier</u> and is denoted by** $\exists$.
- **The <u>uniqueness quantifier</u> is read as <u>unique</u> and is denoted by** $!$.

Example 1.8

If T is a set and $P(x)$ is some statement about x. Then

- $(\forall x \in T)P(x) \Rightarrow$ i.e for all $x \in T, P(x)$ is true.
- $(\exists x \in T)P(x) \Rightarrow$ i.e there exists an $x \in T$ such that $P(x)$ is true.
- $(\exists ! x \in T)P(x) \Rightarrow$ i.e there exists a unique (exactly one) $x \in T$ such that $P(x)$ is true.

Example 1.9

- $(\forall x \in \mathbb{R})(x + x = 2x) \Rightarrow$ i.e. for all $x \in \mathbb{R}$, $x + x = 2x$.
- $(\exists x \in \mathbb{R})(x + 2 = 2) \Rightarrow$ i.e. there exists an $x \in \mathbb{R}$ such that $x + 2 = 2$.
- $(\exists! x \in \mathbb{R})(x + 2 = 2) \Rightarrow$ i.e. there exists a unique (exactly one) $x \in \mathbb{R}$ such that $x + 2 = 2$.

Example 1.10

The following all have the same meaning (and are true)

- for all x and for all y, $(x+y)^2 = x^2 + 2xy + y^2$,
- for all x and y, $(x+y)^2 = x^2 + 2xy + y^2$,
- for each x and each y, $(x+y)^2 = x^2 + 2xy + y^2$,
- $\forall x \forall y (x+y)^2 = x^2 + 2xy + y^2$.

Example 1.11

The following statements all have the same meaning (and are true)

- there exists an irrational number,
- there is at least one irrational number,
- some real number is irrational,
- irrational numbers exists,
- $\exists x$ (x is irrational).

Example 1.12

Goldbach conjecture

Every even natural number greater than 2 is the sum of two primes. This can be written, in logical notation, as

$$\forall n \in \mathbb{N} \backslash \{1\}, \exists p, q, \text{ prime } 2n = p + q$$

which means for all $n \in \mathbb{N}$ which is greater than 1, there exists prime numbers p and q such that $p + q = 2n$.

Note that it is not yet known whether this true or false (an unsolved problem in mathematics).

Order of quantifier

- When you have multiple occurrences of a single quantifier, order does not matter:

$$\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$$
$$\exists x \exists y P(x, y) \Leftrightarrow \exists y \exists x P(x, y)$$

- When you have multiple occurrences of different quantifiers, order does matter:

$$\forall x \exists y P(x, y) \not\Leftrightarrow \exists y \forall x P(x, y)$$

Example 1.13

Consider following two statements:

$$\forall x \exists y (x < y) \tag{1}$$

$$\exists y \forall x (x < y) \tag{2}$$

- We read statement (1) as **for all** x **there exists** y **such that** $x < y$.
- We read statement (2) as **there exists** y **such that for all** x**,** $x < y$.

Example 1.13

Cont...

- Suppose that x and y lie in real line.

- Let x be an arbitrary number. Then $x < x + 1$ and so $x < y$ is true if $y = x + 1$. Hence the statement $\exists y(x < y)$ is true. But x is an arbitrary real number and so the statement **for all** x **there exists** y **such that** $x < y$ **is true**. i.e (1) is true.

- We could justify that the statement (2) is false as follows: Let y be an arbitrary real number. Then $y + 1 < y$ is false. Hence the statement $\forall x(x < y)$ is false. Since y is an arbitrary real number it follows that the statement (2) **there exists** y **such that for all** x**,** $x < y$ **is false**.

Negation of quantifiers

- Now we are concerned on how to develop a rule for **negating proposition** with quantifiers which is one of the important aspects of mathematical analysis.
- This kind of negation will often be used in this course as well.

The negation of universal quantifier
Example 1.14

Let us say, for instance, that **you all are following computer science as a subject**. Symbolically we can formulate this as follows:

- $P(x)$: x is following computer science as a subject.
- $\forall x P(x)$: All are following computer science as a subject.

The negation of universal quantifier

Example 1.14⇒Cont...

- Naturally, you will disagree with $\forall x P(x)$ and some of you will complain.
- So it is natural to aspect the **negation** of the proposition **all are following computer science as a subject** is the proposition **some of you are not following computer science as a subject**.
- Symbolically, the negation of $\forall x P(x)$ is $\exists x \sim P(x)$.
- Note that the proposition $\exists x \sim P(x)$ can also be read as there exist at least one x such that $\sim P(x)$.

The negation of existential quantifier

Example 1.15

1. The negation of "There exists an x such that $P(x)$" is "For all x, not($P(x)$)."

   Symbolically, the negation of $\exists x P(x)$ is $\forall x \sim P(x)$.

2. The negation of "There exists x in T such that $P(x)$" is "For all x in T, not($P(x)$)".

   Symbolically, the negation of $\exists x \in T\ P(x)$ is $\forall x \in T \sim P(x)$.

The negation of existential quantifier
Example 1.16

- The negation of "There exists an x such that $f(x) > 0$", is "For all x, not$[f(x) > 0]$", which can be rephrased in positive form as "For all x, $f(x) \leq 0$".
- Symbolically, the negation of $\exists x f(x) > 0$ is $\forall x f(x) \leq 0$.

Remark 1

- The negation of $\forall x P(x)$, i.e. the statement $\sim(\forall x P(x))$ is equivalent to $\exists x(\sim P(x))$.
- Likewise negation of $\forall x \in \mathbb{R} P(x)$, i.e. the statement $\sim(\forall x \in \mathbb{R} P(x))$ is equivalent to $\exists x \in \mathbb{R}(\sim P(x))$.

Remark 2

- The negation of $\exists x P(x)$, i.e. the statement $\sim (\exists x P(x))$ is equivalent to $\forall x (\sim P(x))$.
- Likewise negation of $\exists x \in \mathbb{R} P(x)$, i.e. the statement $\sim (\exists x \in \mathbb{R} P(x))$ is equivalent to $\forall x \in \mathbb{R} (\sim P(x))$.

Remark 3

- The negation of $\forall x \exists y P(x, y)$ is equivalent to $\exists x \forall y (\sim P(x, y))$.
- The negation of $\exists x \forall y P(x, y)$ is equivalent to $\forall x \exists y (\sim P(x, y))$.

Example 1.17

Suppose that a is a fixed real number. Then the negation of $\forall x \in \mathbb{R}(x > a)$ is $\exists x \in \mathbb{R} \sim (x > a)$ or equivalently $\exists x \in \mathbb{R}(x \leq a)$.

Example 1.18

Suppose that a is a fixed real number. Then the negation of $\exists x \in \mathbb{R}(x > a)$ is $\forall x \in \mathbb{R} \sim (x > a)$ or equivalently $\forall x \in \mathbb{R}(x \leq a)$.

Example 1.19

The **Goldbach conjecture** can be written, in logical notation, as

$$\forall n \in \mathbb{N} \backslash \{1\}, \exists p, q, \text{ prime } 2n = p + q.$$

Write down the negation of **Goldbach conjecture** in logical notation.

The negation of **Goldbach conjecture** is, in logical notation,

$$\exists n \in \mathbb{N}\backslash\{1\}, \forall p, q, \text{ prime } 2n \neq p + q.$$

Example 1.20

For every $\epsilon > 0$ there exists $n \in N$ such that $0 < \frac{1}{n} < \epsilon$.

(a) Write down the statement symbollically.

(b) Write down the negation of the statement symbollically.

Example 1.19
Solution

(a) $\forall \epsilon > 0 \quad \exists n \in \mathbb{N} \quad \frac{1}{n} < \epsilon.$

(b) $\exists \epsilon > 0 \quad \forall n \in \mathbb{N} \quad \frac{1}{n} \geq \epsilon.$

Thank you !