Mathematics for Biology

MAT1142

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Maxima and Minima

Introduction

- When using mathematics to model the physical world in which we live, we frequently express physical quantities in terms of **variables**.
- Then, **functions** are used to describe the ways in which these variables change.
- A scientist or engineer will be interested in the ups and downs of a function, its maximum and minimum values, its turning points.

Maximum and minimum points

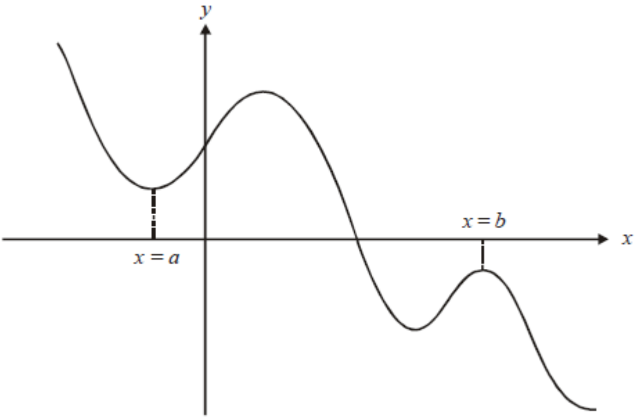


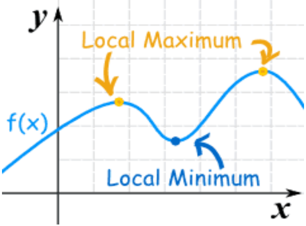
Figure: maximum and minimum points

Locating maximum and minimum points

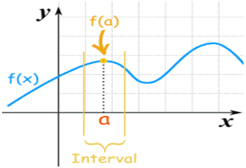
- Drawing a graph of a function using a computer graph plotting package will reveal behaviour of the function.
- But if we want to know the precise location of maximum and minimum points, we need to turn to algebra and differential calculus.
- In this section we look at how we can find maximum and minimum points in this way.

Local maximum and local minimum

- The **local maximum** and **local minimum** (plural: maxima and minima) of a function, are the largest and smallest value that the function takes at a point within a given interval.
- It may not be the minimum or maximum for the whole function, but **locally** it is.



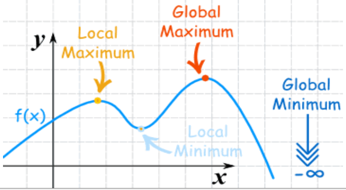
- To define a local maximum, we need to consider an interval.
- Then a **local maximum** is the point where, the height of the function is greater than (or equal to) the height anywhere else in that interval.
- For example, $f(a) \geq f(x)$ for all x in the interval (see the Figure). Therefore, the function f has a local maximum at the point a.



- To define a local minimum, we need to consider an interval.
- Then a **local minimum** is the point where, the height of the function is lowest than (or equal to) the height anywhere else in that interval.
- For example, if $f(a) \leq f(x)$ for all x in the interval, then, the function f has a local minimum at the point a.

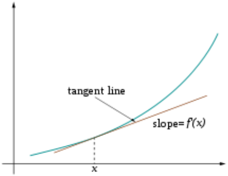
Global maximum and global minimum

- The maximum or minimum over the entire function is called a **global** or **absolute** maximum or minimum.
- There is **only one** global maximum.
- And also there is **only one** global minimum.
- But there can be **more than one** local maximum or minimum.



Stationary points

- The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the **slope of the tangent** to the curve $y = f(x)$ at the point x.
- The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of a function $y = f(x)$ tell us a lot about the shape of a curve.
- The **stationary points** of a function are those points where the derivative of the function is zero.



Stationary points

Types of stationary points

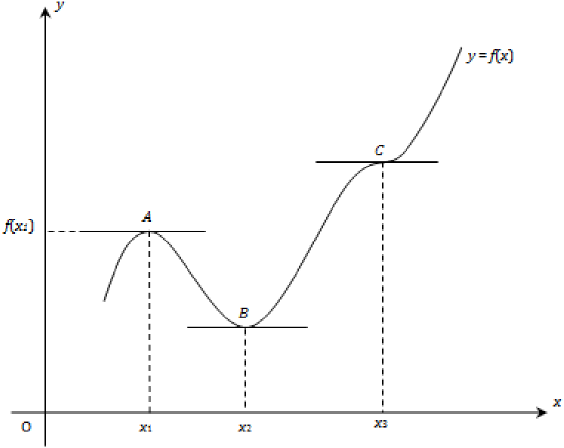


Figure: stationary points

Find the stationary points of the following functions.

(i) $f(x) = 3x^2 + 2x - 9$

(ii) $h(x) = x^3 - 6x^2 + 9x - 2$

(iii) $g(t) = 16 - 6t - t^2$

(iv) $y = 3u^2 - 4u + 7$

Find the stationary points of the following functions.

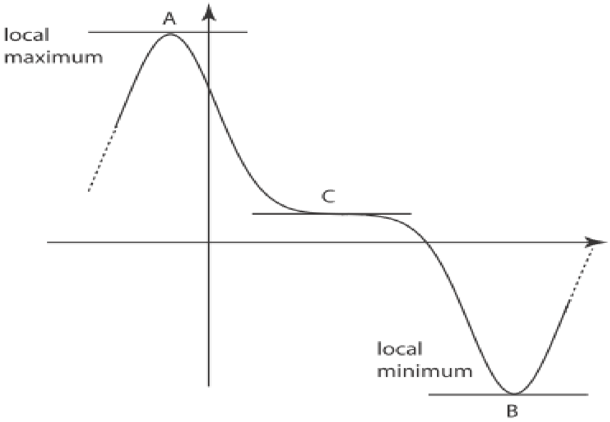
(i) $y = \frac{1}{3}x^3 - x^2 - 3x + 2$

(ii) $y = x^3 - 6x^2 - 15x + 16$

(iii) $f(s) = \frac{1}{3}s^2 - 12s + 32$

(iv) $y = x^3 - 12x + 12$

Classification of stationary points



Classification of stationary points
Cont...

- If we draw tangents to the graph at the points A, B and C, note that these are parallel to the x axis.
- They are horizontal.
- This means that at each of the points A, B and C the slope of the graph is zero.

- We know that the slope of a graph is given by $\mathrm{d}y/\mathrm{d}x$.
- Consequently, $\mathrm{d}y/\mathrm{d}x = 0$ at points A, B and C.
- Therefore, all of these points are stationary points.

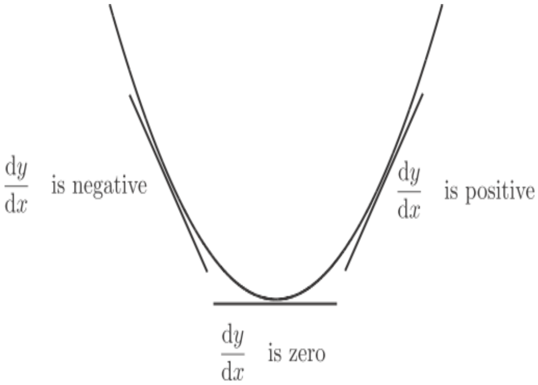
Classification of stationary points
Cont...

- Notice that at points A and B the curve actually turns.
- These two stationary points are referred to as **turning points**.
- Point C is not a turning point and it is an **inflection point**.
- Because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

- So all turning points are stationary points.
- But not all stationary points are turning points.
- In other words, there are points for which $\mathrm{d}y/\mathrm{d}x = 0$ which are not turning points.

- **Point A** is called a **local maximum** because in its immediate area it is the highest point, and so represents the greatest or maximum value of the function.
- **Point B** is called a **local minimum** because in its immediate area it is the lowest point, and so represents the least, or minimum value of the function.
- **Point C** is called a **inflection point**, because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

The graph of a minimum point



The graph of a minimum point

Behavior of $\mathrm{d}y/\mathrm{d}x$ around a minimum point

- Notice that to the left of the minimum point, $\mathrm{d}y/\mathrm{d}x$ is negative because the tangent has negative slope.
- At the minimum point, $\mathrm{d}y/\mathrm{d}x = 0$.
- To the right of the minimum point $\mathrm{d}y/\mathrm{d}x$ is positive, because here the tangent has a positive slope.
- So, $\mathrm{d}y/\mathrm{d}x$ goes from negative, to zero, to positive as x increases.
- In other words, $\mathrm{d}y/\mathrm{d}x$ must be increasing as x increases.

Distinguishing minimum points from stationary points

- In fact, we can use this observation, once we have found a stationary point, to check if the point is a minimum.
- If $\mathrm{d}y/\mathrm{d}x$ is increasing near the stationary point then that point must be minimum.

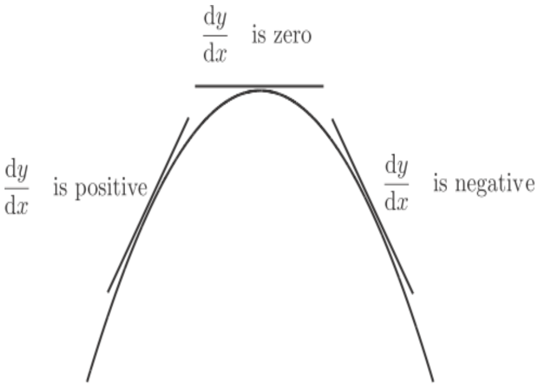
Distinguishing minimum points from stationary points

Behavior of second derivative around a minimum point

- The derivative of $\mathrm{d}y/\mathrm{d}x$ called the **second derivative**, is written $\mathrm{d}^2y/\mathrm{d}x^2$.
- If $\mathrm{d}^2y/\mathrm{d}x^2$ is positive then we will know that $\mathrm{d}y/\mathrm{d}x$ is increasing.
- So we will know that the stationary point is a minimum.

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ there, then that point must be a minimum.

The graph of a maximum point



The graph of a maximum point
Behavior of $\mathrm{d}y/\mathrm{d}x$ around a maximum point

- Notice that to the left of the maximum point, $\mathrm{d}y/\mathrm{d}x$ is positive because the tangent has positive slope.
- At the maximum point, $\mathrm{d}y/\mathrm{d}x = 0$.
- To the right of the maximum point $\mathrm{d}y/\mathrm{d}x$ is negative, because here the tangent has a negative slope.
- So, $\mathrm{d}y/\mathrm{d}x$ goes from positive, to zero, to negative as x increasing.
- In other words, $\mathrm{d}y/\mathrm{d}x$ must be decreasing as x increases.

Distinguishing maximum points from stationary points

- In fact, we can use this observation, once we have found a stationary point, to check if the point is a maximum.
- If $\mathrm{d}y/\mathrm{d}x$ is decreasing near the stationary point then that point must be maximum.

Distinguishing maximum points from stationary points

Behavior of second derivative around a maximum point

- The derivative of $\mathrm{d}y/\mathrm{d}x$ called the **second derivative**, is written $\mathrm{d}^2y/\mathrm{d}x^2$.
- If $\mathrm{d}^2y/\mathrm{d}x^2$ is negative then we will know that $\mathrm{d}y/\mathrm{d}x$ is decreasing.
- So we will know that the stationary point is a maximum.

Distinguishing maximum points from stationary points

Remark

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ at a point, and if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ there, then that point must be a maximum.

Summary of classification of stationary points

- We can locate the positions of stationary points by looking for points where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.
- As we have seen, it is possible that some such points will not be turning points.
- We can calculate $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at each stationary point.

Summary of classification of stationary points
Cont...

- If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive then the stationary point is a **minimum turning point**.
- If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative, then the point is a **maximum turning poin**.
- If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ this second derivative test does not give us useful information.

Example 1

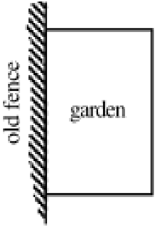
Find the stationary points of the functions and hence determine the nature of these points.

(a) $y = x^2 - 2x + 3$

(b) $y = x^3 - 3x^2 - 9x + 3$

Example 2

Suppose a man needs to fence a rectangular area in his backyard for a garden and he has already fenced one side. So, he only needs to build a new fence along the other 3 sides of the rectangle (see the Figure). If he has 80 feet of fencing available, what dimensions should the garden have in order to enclose the largest possible area?



Example 3
Past paper 2011

A sheet of metal 12 inches by 10 inches is to be used to make a open box. Squares of equal sides x are cut out of each corner and then the sides are folded to make the box (Figure 1). Find the value of x that makes the volume maximum.

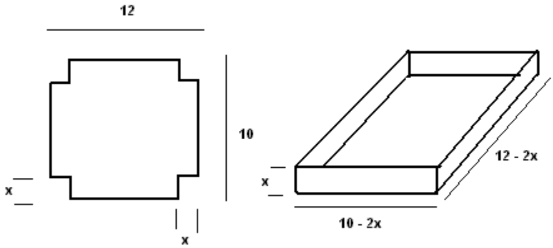
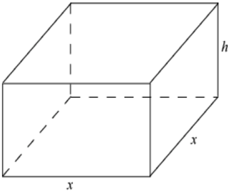


Figure: Used sheet of metal and the open box to be made.

Example 4
Past paper 2012

The material for the square base of a rectangular box with open top costs 27 cents per square *cm* and for the other faces costs $13\frac{1}{2}$ cents per square *cm*. Find the dimensions of such a box of maximum volume which can be made for Rs 65.61.



Example 4
Solution

$$
\begin{aligned}
\text{Cost of making box is} &= 27x^2 + \left(13\frac{1}{2}\right)4xh \\
27x^2 + 54xh &= 6561 \\
x^2 + 2xh &= 243 \\
h &= \frac{243 - x^2}{2x} \qquad (1)
\end{aligned}
$$

We wish to maximize the volume (v)

$$v = x^2 h \qquad (2)$$

From (1) and (2)

$$
\begin{aligned}
v &= x^2\left(\frac{243-x^2}{2x}\right) \\
v &= \frac{1}{2}(243x - x^3) \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{2}(243-3x^2) \\
\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \Rightarrow \frac{1}{2}(243-3x^2) &= 0 \\
243 &= 3x^2 \\
x^2 &= 81 \\
x &= \pm 9
\end{aligned}
$$

x is a lenght. so it cannot be a negative value. So $x = 9$.

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{2}(243 - 3x^2) \\
\frac{\mathrm{d}^2v}{\mathrm{d}x^2} &= \frac{1}{2}(-6x) \\
\frac{\mathrm{d}^2v}{\mathrm{d}x^2} &= -3x \\
\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right)_{(x=9)} &= -3 \times 9 = -27 < 0
\end{aligned}
$$

The second derivative is negative at $x = 9$
Therefore volume becomes maximize when $x = 9$
To find corresponding value of h, we can use (1)

$$\begin{aligned} h &= \frac{243 - x^2}{2x} \\ h &= \frac{243 - 9^2}{2 \times 9} \\ h &= 9 \end{aligned}$$

Exercise

Determine the nature of the stationary points of the following functions.

(a) $y = 16 - 6u - u^2$

(b) $y = 3x^2 - 4x + 7$

Thank You