Mathematics for Biology

MAT1142

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Partial Differentiation and Total Differentiation

Partial Differentiation

Why do we need partial derivatives?

Suppose a boy throws a ball vertically upward with a speed of $20ms^{-1}$ and the height of the ball, in meters, after t seconds is approximately given by,

$$H(t) = 20t - 10t^2.$$

- Here, time t is a variable.
- That is the only variable within the given expression.
- So, the height H is a function of one variable.

Why do we need partial derivatives?
Cont...

- Then, $\frac{dH}{dt}$ represents the rate of change of H or velocity v.
- It implies that the rate of change of a function with one variable was shown to be measured by its **derivative**.
- But many applications require functions with more than one variable.
- Can we find derivative of a function with more than one variable?
- To answer that we need the concept of **partial derivatives**.

What are partial derivatives?

- The ideal gas law can be expressed as,
$$PV = kT.$$
- Where P is the pressure, V is the volume, T is the absolute temperature of the gas, and k is a constant.
- We can rearranging above equation as,
$$P = kT/V.$$
- It shows that P is a function of two variables T and V.

What are partial derivatives?
Cont...

- If one of the variables, say T, is kept fixed and V changes, then the derivative of P with respect to V measures the rate of change of pressure with respect to volume.
- In this case, it is called the **partial derivative** of P with respect to V and written as, $\frac{\partial P}{\partial V}$.

What is partial differentiation?

The process of differentiating a function with respect to one variable, while treating the other variables, as a constant is called **partial differentiation**.

Example 1

If $P = \frac{kT}{V}$, find the partial derivatives of P,

(a) with respect to T.

(b) with respect to V.

Example 1
Solution

(a) If we treated V as a constant, then

$$\begin{aligned} P &= \frac{kT}{V} \\ \frac{\partial P}{\partial T} &= \frac{k}{V}. \end{aligned}$$

(b) If we treated T as a constant, then

$$\begin{aligned} P &= \frac{kT}{V} = kTV^{-1} \\ \frac{\partial P}{\partial V} &= -kTV^{-2} \\ &= \frac{-kT}{V^2}. \end{aligned}$$

Example 2

If $z = f(x, y) = 2x^2y + 3xy^2$, find the partial derivatives of z,

(a) with respect to x.

(b) with respect to y.

Example 2
Solution

(a)

- The rate of change of z with respect to x may be found out when y is constant.
- To do this, z is differentiated with respect to x, treating y as a constant.

$$\begin{aligned} z &= 2x^2y + 3xy^2 \\ \frac{\partial z}{\partial x} &= 4xy + 3y^2 \end{aligned}$$

- The quantity $4xy + 3y^2$ is called the partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$.

Example 2
Solution

(b)

- The rate of change of z with respect to y may be found out when x is constant.
- To do this, z is differentiated with respect to y, treating x as a constant.

$$\begin{aligned} z &= 2x^2y + 3xy^2 \\ \frac{\partial z}{\partial y} &= 2x^2 + 6xy \end{aligned}$$

- The quantity $2x^2 + 6xy$ is called the partial derivative of z with respect to y and is denoted by $\frac{\partial z}{\partial y}$.

Example 3

(i) $f(x, y) = 3x + 5y$

(ii) $z = x^3 + y^2$

(iii) $g(x, y) = 2x^2y^2 + y^2x^4$

(iv) $w = x^2y + y^2x + 8$

(v) $z = x^3y + e^x$

(vi) $h(x, y) = \dfrac{x - y}{x + y}$

(vii) $u = \cos(x^2y)$

(viii) $L = 2x\sin(x^2y)$

Example 4
Past paper 2012

Suppose in a flat metal plate the temperature at a point (x, y) varies according to position. Let the temperature at a point (x, y) be given by

$$T(x, y) = 50/(1 + x^2 + y^2);$$

where T is measured in Celsius and x and y in meters.
What is the rate of change of temperature with respect to distance at the point (3,2)

(i) in the x-direction?

(ii) in the y-direction ?

Example 4
Solution

(i)

$$
\begin{aligned}
T(x,y) &= \frac{50}{1+x^2+y^2} \\
&= 50(1+x^2+y^2)^{-1} \\
\frac{\partial T}{\partial x} &= 50(-1)(1+x^2+y^2)^{-2}(2x) \\
\frac{\partial T}{\partial x} &= \frac{-100x}{(1+x^2+y^2)^2} \\
\left(\frac{\partial T}{\partial x}\right)_{(3,2)} &= \frac{-100(3)}{(1+3^2+2^2)^2} \\
&= -1.5306cm^{-1}
\end{aligned}
$$

Example 4
Solution

(ii)

$$
\begin{aligned}
T(x,y) &= \frac{50}{1+x^2+y^2} \\
&= 50(1+x^2+y^2)^{-1} \\
\frac{\partial T}{\partial y} &= 50(-1)(1+x^2+y^2)^{-2}(2y) \\
\frac{\partial T}{\partial y} &= \frac{-100y}{(1+x^2+y^2)^2} \\
\left(\frac{\partial T}{\partial y}\right)_{(3,2)} &= \frac{-100(2)}{(1+3^2+2^2)^2} \\
&= -1.0204cm^{-1}
\end{aligned}
$$

Exercise

(i) $f(x, y) = 7x + 6y$

(ii) $z = 2x^4 + 3y^5$

(iii) $f(x, y) = 5x^5y^2 + y^3x$

(iv) $f(x, y) = y^2x + 5x$

(v) $f(x, y) = x^3y^4 + 2e^y$

(vi) $w = \dfrac{x}{y}$

(vii) $z = \sin(xy^2)$

(viii) $f(x, y) = 5x\sin(xy^2)$

Partial derivatives with more than two variables

- Suppose z is a function of independent variables, $x_1, x_2, ..., x_n$.
- Then, z can be differentiated with respect to x_1 when other variables $x_2, ..., x_n$ are kept constant.
- The partial derivative of z with respect to x_1 is denoted as $\frac{\partial z}{\partial x_1}$.
- Similarly other partial derivatives are,

$$\frac{\partial z}{\partial x_2}, \frac{\partial z}{\partial x_3}, \frac{\partial z}{\partial x_4}, ..., \frac{\partial z}{\partial x_n}.$$

Example

Suppose that we have the function

$$w = f(x, y, z) = x\sin(yz) + ye^{x} + 5y^{4}.$$

Then find, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, and $\frac{\partial w}{\partial z}$.

Total Differentiation

Total differential for function of two variables

If we consider z as a function of two variables such as $z = f(P, T)$, then the total differential $\mathrm{d}z$ can be written as,

$$\mathrm{d}z = \left(\frac{\partial z}{\partial P}\right)_T \mathrm{d}P + \left(\frac{\partial z}{\partial T}\right)_P \mathrm{d}T.$$

Total differential for function of three variables

If we consider z as a function of three variables such as $z = f(P, V, T)$, then the total differential $\mathrm{d}z$ can be written as,

$$\mathrm{d}z = \left(\frac{\partial z}{\partial P}\right)_{V,T} \mathrm{d}P + \left(\frac{\partial z}{\partial V}\right)_{P,T} \mathrm{d}V + \left(\frac{\partial z}{\partial T}\right)_{P,V} \mathrm{d}T.$$

Where $\left(\frac{\partial z}{\partial P}\right)_{V,T}$ means that the function z is differentiated with respect to the variable P, keeping the other variables V, T as constants.

Total differential for function of more variables

If we consider z as a function of more variables such as $z = f(a, b, c, d, ...)$, then the total differential $\mathrm{d}z$ can be written as,

$$\mathrm{d}z = \left(\frac{\partial z}{\partial a}\right)_{b,c,d,...} \mathrm{d}a + \left(\frac{\partial z}{\partial b}\right)_{a,c,d,...} \mathrm{d}b + \left(\frac{\partial z}{\partial c}\right)_{a,b,d,...} \mathrm{d}c + ...$$

Where $\left(\frac{\partial z}{\partial a}\right)_{b,c,d,...}$ means that the function z is differentiated with respect to the variable a, keeping the other variables $b, c, d, ...$ as constants.

Example 1

Find the total differential of

(i) $u = 3p^2 + 4T^3$.

(ii) $v = 3p^2 + 4T^2 + P^2T^4$.

Example 2
Past paper 2012

If $p = \ln(e^v + e^q)$, find the total differential of p.

Example 3

Cobb-Douglas production function is given by

$$Y = f(K, L) = AK^{\alpha}L^{1-\alpha},$$

where A and α are constants. Find the total differential of Y.

A function $f(x_1, x_2, ..., x_m)$ is said to be homogeneous of degree n with the variables $x_1, x_2, ..., x_m$ if

$$f(\lambda x_1, \lambda x_2, ..., \lambda x_m) = \lambda^n f(x_1, x_2, ..., x_m)$$

Example 1

Find out whether following equations are homogeneous or not? If homogeneous, what is the degree of it?

(i) $f(x, y, z) = x^2 + y^2 + z^2$.

(ii) $g(x, y) = x + y$.

(iii) $h(x, y) = x^2 + 3y^2 - 6xy$.

Example 2
Past paper 2012

The ideal gas equation is given by,

$$f(P, V, T) = \frac{PV}{T}.$$

State whether this function is homogeneous or not?
If it is homogeneous, what is the degree of it?

Example 3

State whether $f(x, y) = x^2 + 3y^2 - 6xy + 5$ is homogeneous or not?

Euler's theorem on homogeneous functions

If $f(x_1, x_2, ..., x_m)$ is a homogeneous function of degree n with the variables $x_1, x_2, ..., x_m$, then we can prove that,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + ... + x_m \frac{\partial f}{\partial x_m} = nf$$

This result is known as the **Euler's theorem** on homogeneous functions.

Past paper 2011

(a) The van der Walls equation is given in usual notation as

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT;$$

where a and b are constants.

(i) Express P in terms of V, T, a, b and R.

(ii) Determine $\frac{\partial P}{\partial T}$ and $\frac{\partial P}{\partial V}$.

(iii) Write down expression for $\mathrm{d}p$.

(iv) Express $\mathrm{d}p$ when $T = 300$, $V = 2$, $a = 5.5$, $b = 0.035$ and $R = 1.4$.

(v) If T and V could change from 2 and 0.01 respectively, what would be the value of $\mathrm{d}p$?

(b) If $p = \ln(e^{3x} + e^{3y})$ show that $\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y} = 3$.

(c) For reversible, adiabatic expansion of an ideal gas,

$$f(T, V) = TV^{\gamma - 1} \text{ where } \gamma = \frac{C_P}{C_V}.$$

Find whether this equation is homogeneous or not? If it is homogeneous, what is the degree of it?

Thank You