Mathematics for Biology

MAT1142

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Differentiation

Pre-Requisities for Differentiation

Terms and polynomials

- A **term** is an algebraic expression that is either a constant or a product of a constant and one or more variables raised to whole-number powers.
- Examples of terms include: $8,\ 6x^4,\ 5xy^3$.
- A **polynomial** is a finite sum of one or more terms.
- Examples of polynomials are: $6y^4,\ x+13,\ 7x^2-12xy+8y^2$.

The degree of terms and the degree of polynomials

- The **degree of a term** is the sum of the exponents of its variables.
- The degree of a nonzero constant is zero.
- The **degree of a polynomial** is the highest degree of any of its terms.

(a) $3x^5 - 5x^3 - 7x + 13$

This polynomial has four terms, including a fifth-degree term, a third-degree term, a first-degree term, and a constant term. This is a fifth-degree polynomial.

(b) $6x^4 + 4x^2 + x$

This polynomial has three terms, including a fourth-degree term, a second-degree term, and a first-degree term. There is no constant term. This is a fourth-degree polynomial.

General form of a univariate polynomial of degree n

- The simplest polynomials have one variable.
- A one-variable (univariate) polynomial of degree n has the following form:

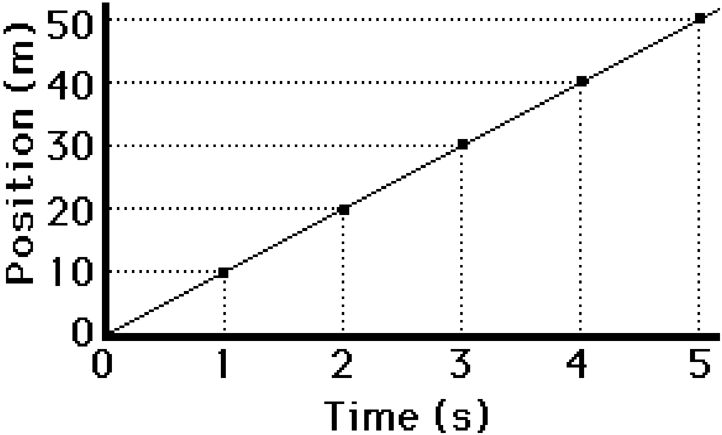
$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 x^0.$$

- Where the a's represent the coefficients and x represents the variable.
- Because $x^1 = x$ and $x^0 = 1$ for all complex numbers x, the above expression can be simplified to:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0.$$

How to calculate the slope of a line?

Let's begin by considering the position versus time graph below.



How to calculate the slope of a line?
Cont...

- The line is sloping upwards to the right.
- But mathematically, by how much does it slope upwards for every 1 second along the horizontal (time) axis?
- To answer this question we must use the slope equation.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}.$$

How to calculate the slope of a line?
Cont...

- Pick two points on the line and determine their coordinates.
- Determine the difference in y-coordinates of these two points (rise).
- Determine the difference in x-coordinates for these two points (run).
- Divide the difference in y-coordinates by the difference in x-coordinates (rise/run).

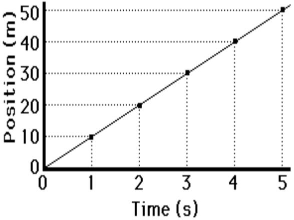
How to calculate the slope of a line?
Example

Find the slop of the line using following points.

(a) For points $(5s, 50m)$ and $(0s, 0m)$.

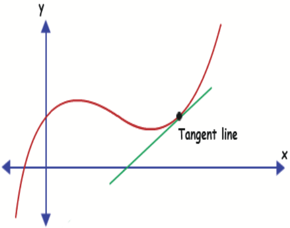
(b) For points $(5s, 50m)$ and $(2s, 20m)$.

(c) For points $(4s, 40m)$ and $(3s, 30m)$.



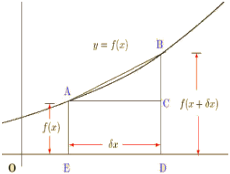
How to calculate the slope of a curve?

- To find slope of a curve, we have to consider **tangents** drawn to the curve at different points.
- Tangent means "a straight line that touches a curve at a point but does not intersect it at that point".



How to calculate the slope of a curve?
Cont...

- Let $y = f(x)$ is a function as shown in Figure.
- The straight line AB has slope BC/CA.
- As the point B moves along the curve toward A, the straight line AB tends toward the tangent to the curve at A.
- At the same time, the value of the slope BC/CA tends toward the slope of the tangent to the curve at A.



How to calculate the slope of a curve?
Cont...

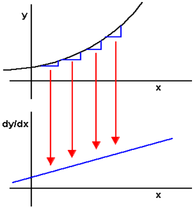
$$
\begin{aligned}
\frac{BC}{CA} &= \frac{f(x+\delta x) - f(x)}{\delta x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}
\end{aligned}
$$

The limit, $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$, is called the **derivative** of the function f(x) at the point $A(x, y)$. Its value is the **slope of the tangent** to the curve at the point $A(x, y)$.

Introduction to Differentiation

Introduction

- **Differentiation** is concerned with the **rate of change** of one quantity with respect to another quantity.
- When illustrating a function on a graph the rate of change is represented by the slope of a tangent.
- So, differentiation is the problem of finding the slope of a tangent at a specific point on a graph.



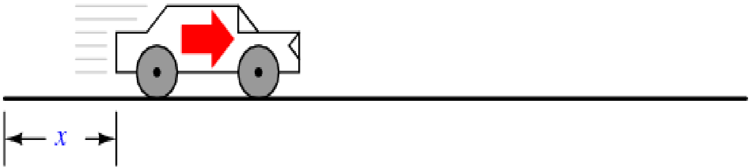
- The instantaneous rate of change of a function $y = f(x)$ at a point $x = a$ is called the **derivative** of $f(x)$ at $x = a$, denoted by $f'(a)$ or $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=a}$.

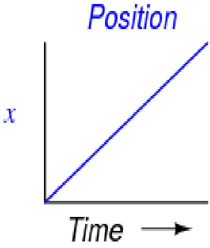
Motivating example 1

- Suppose we were to measure the position of a car, traveling in a direct path (no turns), from its starting point.
- Let us call this measurement, x.
- If the car moves at a rate such that its distance from "start" increases steadily over time, its position will plot on a graph as a linear function (straight line).

Motivating example 1

Position of the car



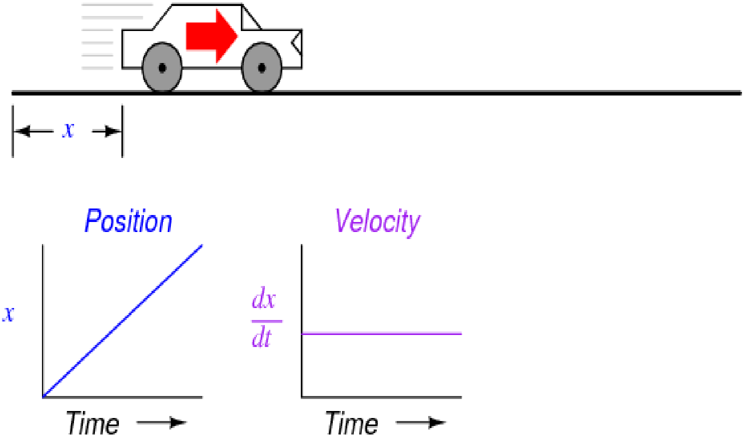


Motivating example 1
Cont...

- The instantaneous rate of change of the car's position with respect to time is called the derivative of the car's position with respect to time.
- The derivative of the car's position with respect to time represents the car's **velocity**.

Motivating example 1

Velocity of the car



- The instantaneous rate of change of the function $x(t)$ at a point $t = a$ is called the **derivative** of $x(t)$ at $t = a$, denoted by $x'(a)$ or $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t=a}$.
- $x'(a)$ is the velocity of the car at $t = a$.

Motivating example 2

A boy throws a ball vertically upward with a speed of $20ms^{-1}$ then the height of the ball, in meters, after t seconds is approximately,

$$H(t) = 20t - 10t^2.$$

Find the average speed of the ball during the following time intervals.

(a) $t = 0.5s$ to $t = 1s$,
(b) $t = 0.5s$ to $t = 0.75s$,
(c) $t = 0.5s$ to $t = 0.6s$,
(d) $t = 0.5s$ to $t = 0.502s$,
(e) $t = 0.5s$ to $t = 0.501s$,
(f) $t = 0.5s$ to $t = 0.5001s$.

The average speed is given by,

$$\text{Average speed} = \frac{\text{Travelled distance}}{\text{Time taken}}.$$

(a) The average speed from $t = 0.5s$ to $t = 1s$ is:

$$\begin{aligned}
&= \frac{\text{Travelled distance}}{\text{Time taken}}\\
&= \frac{H(1) - H(0.5)}{1 - 0.5}\\
&= \frac{[20 \times 1 - 10 \times (1)^2] - [20 \times 0.5 - 10 \times (0.5)^2]}{1 - 0.5}\\
&= 5ms^{-1}
\end{aligned}$$

(b) The average speed from $t = 0.5s$ to $t = 0.75s$ is:

$$
\begin{aligned}
&= \frac{\text{Travelled distance}}{\text{Time taken}} \\
&= \frac{H(0.75) - H(0.5)}{0.75 - 0.5} \\
&= \frac{[20 \times 0.75 - 10 \times (0.75)^2] - [20 \times 0.5 - 10 \times (0.5)^2]}{0.75 - 0.5} \\
&= 7.5ms^{-1}
\end{aligned}
$$

- If we calculate average speeds for each of the above time intervals, a good choice for the speed of the ball at $t = 0.5$ is corresponding to the solution of part (f).
- The next example gives the **general solution** to this problem.

Motivating example 3

If, as in motivative example 2, the height of a ball at time t is given by

$$H(t) = 20t - 10t^2,$$

then find the following :

(a) the average speed of the ball over the time interval from t to $t + \delta t$,

(b) the limit of this average as $\delta t \to 0$.

(a) The height at time $t+\delta t$ is $H(t+\delta t)$ and the height at time t is $H(t)$. The difference in heights is $H(t+\delta t)-H(t)$ and the time interval is δt.

$$
\begin{aligned}
H(t+\delta t)-H(t) &= \left[20(t+\delta t)-10(t+\delta t)^2\right]-\left[20t-10t^2\right] \\
&= 20\delta t-20t\delta t-10(\delta t)^2 \\
&= \delta t\left[20-20t-10\delta t\right]
\end{aligned}
$$

Motivating example 3

Solution⇒Cont...

The required average speed of the ball from t to $t + \delta t$ is:

$$\begin{aligned}
&= \frac{\text{Travelled distance}}{\text{Time taken}} \\
&= \frac{H(t+\delta t) - H(t)}{\delta t} \\
&= \frac{\delta t\,[20 - 20t - 10\delta t]}{\delta t} \\
&= 20 - 20t - 10\delta t.
\end{aligned}$$

Motivating example 3

Solution⇒Cont...

(b)

- As δt gets smaller, i.e. $\delta t \to 0$, the last term becomes negligible.
- The instantaneous speed at time t is $20 - 20t$.
- That is the speed of the ball at time t.

Remark

The speed $v(t)$ is obtained from the height $H(t)$ as,

$$v(t) = H'(t) = \frac{\mathrm{d}H}{\mathrm{d}t} = \lim_{\delta t \to 0} \left[\frac{H(t + \delta t) - H(t)}{\delta t} \right].$$

Steps in differentiating a function using definition

1. Let $y = f(x)$.
2. Then $y + \delta y = f(x + \delta x)$.
3. $y + \delta y - y = f(x + \delta x) - f(x) \Rightarrow \delta y = f(x + \delta x) - f(x)$.
4. $\dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$.
5. $\displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right)$.
6. $\displaystyle\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right)$.

Rule 1 : The derivative of a constant

The derivative of a constant is zero. That is if $y = c$, where c is a constant, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Find the derivatives of the followings.

(i) $y = 7$.

(ii) $y = 7a$ where a is a constant.

(iii) $y = \dfrac{5a}{b^6}$ where a, b are constants.

(iv) $y = a^3 b + b^5 c + 65c^5$ where a, b, c are constants.

(v) $y = \log \sqrt{5} a^3 b^7$ where a and b are constants.

Using the basic definition, find the derivatives of the following functions with respect to x.

(i) $y = x$.

(ii) $y = x^2$.

(iii) $y = x^3$.

(i)

$$
\begin{aligned}
y &= x \\
y + \delta y &= x + \delta x \\
y + \delta y - y &= x + \delta x - x \\
\delta y &= \delta x \\
\frac{\delta y}{\delta x} &= \frac{\delta x}{\delta x} \\
\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1
\end{aligned}
$$

(ii)

$$
\begin{aligned}
y &= x^2\\
y+\delta y &= (x+\delta x)^2\\
y+\delta y - y &= x^2 + 2x\,\delta x + (\delta x)^2 - x^2\\
\delta y &= 2x\,\delta x + (\delta x)^2\\
\frac{\delta y}{\delta x} &= \frac{2x\,\delta x + (\delta x)^2}{\delta x} = 2x + \delta x\\
\lim_{\delta x\to 0}\frac{\delta y}{\delta x} &= \lim_{\delta x\to 0} 2x + \lim_{\delta x\to 0}\delta x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x
\end{aligned}
$$

(iii)

$$\begin{aligned}
y &= x^3 \\
y + \delta y &= (x + \delta x)^3 \\
y + \delta y - y &= x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3 - x^3 \\
\delta y &= 3x^2\,\delta x + 3x(\delta x)^2 + (\delta x)^3 \\
\frac{\delta y}{\delta x} &= \frac{3x^2\,\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x} \\
\frac{\delta y}{\delta x} &= 3x^2 + 3x(\delta x) + (\delta x)^2 \\
\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} 3x^2 + \lim_{\delta x \to 0} 3x(\delta x) + \lim_{\delta x \to 0} (\delta x)^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2
\end{aligned}$$

Differentiating functions using definition

Examples⇒Generalization of results

Function y	Derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$	Can arrange as
x	1	$1x^{1-1}$
x^2	$2x$	$2x^{2-1}$
x^3	$3x^2$	$3x^{3-1}$
x^4	$4x^3$	$4x^{4-1}$
x^5	$5x^4$	$5x^{5-1}$
.	.	.
.	.	.
.	.	.
x^n	nx^{n-1}	nx^{n-1}

Rule 2: The general power rule

If the function is given by $y = x^n$, where n is any positive or negative integer, then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}.$$

Find the derivatives of the followings.

(i) $y = x^7$.

(ii) $y = x^{50}$.

(iii) $y = x^{-6}$.

(iv) $y = \dfrac{1}{x^7}$.

(v) $y = \dfrac{1}{t}$.

(vi) $y = \dfrac{1}{u^{-15}}$.

Using the basic definition, find the derivatives of the following functions with respect to x.

(i) $y = ax$.

(ii) $y = ax^2$.

(iii) $y = ax^3$.

Differentiating a constant times a function using definition

Examples ⇒ Solution

(i)

$$
\begin{aligned}
y &= ax \\
y + \delta y &= a(x + \delta x) \\
y + \delta y - y &= ax + a\,\delta x - ax \\
\delta y &= a\,\delta x \\
\frac{\delta y}{\delta x} &= \frac{a\,\delta x}{\delta x} \\
\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} a \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= a
\end{aligned}
$$

(ii)

$$
\begin{aligned}
y &= ax^2 \\
y + \delta y &= a(x + \delta x)^2 \\
y + \delta y - y &= ax^2 + 2ax\,\delta x + a(\delta x)^2 - ax^2 \\
\delta y &= 2ax\,\delta x + a(\delta x)^2 \\
\frac{\delta y}{\delta x} &= \frac{2ax\,\delta x + a(\delta x)^2}{\delta x} = 2ax + a\,\delta x \\
\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} 2ax + \lim_{\delta x \to 0} a\,\delta x \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2ax
\end{aligned}
$$

(iii)

$$
\begin{aligned}
y &= ax^3 \\
y + \delta y &= a(x+\delta x)^3 \\
y + \delta y - y &= ax^3 + 3ax^2\,\delta x + 3ax\,(\delta x)^2 + a(\delta x)^3 - ax^3 \\
\delta y &= 3ax^2\,\delta x + 3ax(\delta x)^2 + a(\delta x)^3 \\
\frac{\delta y}{\delta x} &= \frac{3ax^2\,\delta x + 3ax(\delta x)^2 + a(\delta x)^3}{\delta x} \\
\frac{\delta y}{\delta x} &= 3ax^2 + 3ax(\delta x) + a(\delta x)^2 \\
\lim_{\delta x \to 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \to 0} 3ax^2 + \lim_{\delta x \to 0} 3ax(\delta x) + \lim_{\delta x \to 0} a(\delta x)^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3ax^2
\end{aligned}
$$

Differentiating a constant times a function using definition
Examples⇒Generalization of results

Function y	Derivative $\mathrm{d}y/\mathrm{d}x$	Can arrange as
ax	a	ax^{1-1}
ax^2	$2ax$	$2ax^{2-1}$
ax^3	$3ax^2$	$3ax^{3-1}$
ax^4	$4ax^3$	$4ax^{4-1}$
ax^5	$5ax^4$	$5ax^{5-1}$
.	.	.
.	.	.
.	.	.
ax^n	nax^{n-1}	nax^{n-1}

Rule 3: The derivative of a constant times a function

If the function is given by $y = ax^n$, where a is any real number and n is any positive or negative integer, then:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}.$$

Find the derivatives of the followings.

(i) $y = 5x^6$.

(ii) $y = -\frac{1}{4}x^8$.

(iii) $y = 5u^{-8}$.

(iv) $y = \frac{4}{t^5}$.

(v) $y = -\frac{13}{x^7}$.

(vi) $y = \frac{7x^3}{x^{-15}}$.

More examples

Find the derivatives of the following functions.

(i) $y = \sqrt{x}$.

(ii) $y = 7\dfrac{1}{\sqrt{x}}$.

(iii) $y = \sqrt[5]{u}$.

(iv) $y = \sqrt{p^5}$.

Rule 4: The derivative of a sum or a difference

- If $y = h(x) + g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}.$$

- If $y = h(x) - g(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}x} - \frac{\mathrm{d}g}{\mathrm{d}x}.$$

Rule 4: The derivative of a sum or a difference

Proof

$$
\begin{aligned}
y &= h(x) + g(x) \\
y + \delta y &= h(x + \delta x) + g(x + \delta x) \\
y + \delta y - y &= h(x + \delta x) + g(x + \delta x) - (h(x) + g(x)) \\
\delta y &= h(x + \delta x) + g(x + \delta x) - h(x) - g(x) \\
\delta y &= h(x + \delta x) - h(x) + g(x + \delta x) - g(x)
\end{aligned}
$$

Rule 4: The derivative of a sum or a difference
Proof⇒Cont...

$$
\begin{aligned}
\frac{\delta y}{\delta x} &= \frac{h(x+\delta x)-h(x)+g(x+\delta x)-g(x)}{\delta x} \\
\frac{\delta y}{\delta x} &= \frac{h(x+\delta x)-h(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x} \\
\lim_{\delta x\to 0}\frac{\delta y}{\delta x} &= \lim_{\delta x\to 0}\frac{h(x+\delta x)-h(x)}{\delta x}+\lim_{\delta x\to 0}\frac{g(x+\delta x)-g(x)}{\delta x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}h}{\mathrm{d}x}+\frac{\mathrm{d}g}{\mathrm{d}x}
\end{aligned}
$$

Find the derivatives of the following functions.

(i) $y = 5x^6 + 3x^5$.

(ii) $y = -\frac{1}{4}x^7 + 5x$.

(iii) $y = 7x^{-5} + 3x^5$.

(iv) $y = \frac{2}{x^5} + \frac{2}{5}$.

(v) $y = -\frac{9}{x^5} - 3x^7$.

(vi) $y = \frac{5x^3}{x^{-15}} + x^{\frac{2}{3}}$.

Rule 5: The derivative of a polynomial function

- Let $y = f(x)$ be a polynomial function.
- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x)$.
- $f'(x)$ is called the derivative of the polynomial $f(x)$.

Rule 5: The derivative of a polynomial function

The derivative of a univariate polynomial of degree n

$$
\begin{aligned}
y &= f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= f'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{(n-1)-1} + \ldots + a_2 2 x^{2-1} \\
&+ a_1 x^{1-1} + 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= f'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + \ldots + a_2 2x + a_1.
\end{aligned}
$$

Find the derivatives of the following functions. Suppose n is a constant.

(i) $y = 3x^{-7} + 5x^4 + 7x + 5.$

(ii) $y = 8 + \frac{3}{x} - \frac{7}{x^2}.$

(iii) $y = \frac{1}{4}x^4 + \frac{1}{8}x + \frac{1}{7}.$

(iv) $y = 3t^3 + 6t^2 + 9t + 7.$

(v) $v = 3x^n - nx^6 + 7n.$

(vi) $h = 2u^3 + \frac{1}{u^2} + 7u + 3.$

Rule 6: The derivatives of trigonometric functions

y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$

Find the derivatives of the following functions.

(i) $y = 5x^3 + \sin x$.

(ii) $y = \tan x + 2\cos x + 7x^2 + 4$.

(iii) $y = \cot x + \dfrac{1}{x^7} + \dfrac{1}{\cos x}$.

(iv) $y = 5\cos t + 2t^5 + 4t + \dfrac{1}{t^7}$.

(v) $v = 5x^3 + \dfrac{1}{\sin x}$.

(vi) $h = \cos x \tan x + 5x^2 \csc x \sin x$.

Rule 7: The derivative of exponential function

$$
\begin{aligned}
y &= e^x \\
y &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \ldots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 + 1 + \frac{2x}{1 \times 2} + \frac{3x^2}{1 \times 2 \times 3} + \frac{4x^3}{1 \times 2 \times 3 \times 4} \\
&+ \frac{5x^4}{1 \times 2 \times 3 \times 4 \times 5} + \ldots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x
\end{aligned}
$$

Find the derivatives of the following functions.

(i) $y = 3e^x + 7x$.

(ii) $y = \tan x + \sin x + 4x^5 + 7e^x$.

(iii) $y = \dfrac{e^x}{5} + \dfrac{1}{x^3}$.

(iv) $y = 7e^t + t^5 + 2(t + e^t)$.

(v) $v = 4x^3 + \dfrac{1}{e^{-x}}$.

(vi) $h = \dfrac{4}{e^{-x}} + \dfrac{2}{\tan x}$.

Rule 8: The derivative of a product

The derivative of the product $y = u(x)v(x)$, where u and v are both functions of x is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u \times \frac{\mathrm{d}v}{\mathrm{d}x} + v \times \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Find the derivatives of the following functions.

(i) $y = (x+3)(x+2)$.

(ii) $y = x^2 \sin x$.

(iii) $y = \sin x \cos x$.

(iv) $y = e^x(5x^3 - 1)$.

(v) $p = (x^3 + 5x)(x^5 - 7x)$.

(vi) $h = \sin^2 x \csc x \tan x$.

(vii) $y = x^3 e^x \cos x$.

(viii) $y = (x+3)(\sin x)e^x$.

Rule 9: The derivative of a quotient

The derivative of the quotient $y = u(x)/v(x)$, where u and v are both function of x is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v \times \frac{\mathrm{d}u}{\mathrm{d}x} - u \times \frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

Find the derivatives of the following functions.

(i) $y = \dfrac{(x+3)}{(x+5)}$.

(ii) $y = \dfrac{(x^3-1)}{5x^2}$.

(iii) $y = \dfrac{x^5+5x^2-4}{x^2-1}$.

(iv) $y = \dfrac{\sin t + t}{\cos t}$.

(v) $p = \dfrac{e^x}{\cos x}$.

(vi) $h = \dfrac{(x^2+5)}{5e^x - 2\tan x}$.

Rule 10: The derivative of a function of function

If y is a function of u, i.e. $y = f(u)$, and u is a function of x, i.e. $u = g(x)$ then the derivative of y with respect to x is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \text{ (The chain rule).}$$

Find the derivatives of the following functions.

(i) $y = (x+3)^5$.

(ii) $y = (7x^5 - 3x^2 - 2x + 9)^4$.

(iii) $y = (2x^4 + 6)^7$.

Find the derivatives of the following functions.

(i) $y = (x+3)^5$.

(ii) $y = (7x^5 - 3x^2 - 2x + 9)^4$.

(iii) $y = (2x^4 + 6)^7$.

(iv) $y = \sqrt{1 + x^2}$.

(v) $y = \cos(x^2)$.

(vi) $y = \sin(x^2 + 3x + 5)$.

(vii) $y = e^{3x}$.

(viii) $y = e^{x^2}$.

(ix) $y = e^{x^3+2x+7}$.

(x) $y = \dfrac{1}{\sqrt{x^2+1}}$.

Motivating example 2 (Cont...)

- The height $H(t)$ in metres of a ball thrown vertically at $20ms^{-1}$, was given by,

$$H(t) = 20t - 10t^2.$$

- The **velocity** of the ball, $v\ ms^{-1}$, after t seconds, was given by,

$$v(t) = \frac{\mathrm{d}H}{\mathrm{d}t} = 20 - 20t.$$

Motivating example 2 (Cont...)

- The rate of change of velocity with time, which is the **acceleration**, is then given by $a(t)$, where,

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -20ms^{-2}.$$

Motivating example 2 (Cont...)

- The acceleration was derived from $H(t)$ by two successive differentiations.
- The resulting function, which in this case is $a(t)$, is called the second derivative of $H(t)$ with respect to t.
- It can be written mathematically as,

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}H}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 H}{\mathrm{d}t^2}.$$

The second derivative

- The second derivative is the derivative of the derivative of a function.
- The derivative of the function $f(x)$ may be denoted by $f'(x)$, and its second derivative is denoted by $f''(x)$.

Higher derivatives

- The third derivative is the derivative of the derivative of the derivative of a function, which can be represented by $f'''(x)$.
- This is read as "the third derivative of $f(x)$".
- This can continue as long as the resulting derivative is itself differentiable, with the fourth derivative, the fifth derivative, and so on.
- Any derivative beyond the first derivative can be referred to as a higher derivative.

Examples

Find the first and the second derivatives of the following functions.

(i) $f(x) = x^3$

(ii) $g(x) = x^3 - 6x^2 + 9x - 2$

(iii) $y = x^3 + e^x$

Thank You