Mathematics for Biology

MAT1142

Department of Mathematics
University of Ruhuna

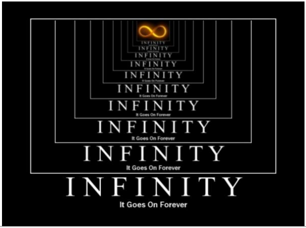
A.W.L. Pubudu Thilan

Limits

Pre-Requisities for Limits

What is infinity?

- Infinity is the idea of something that has no end.
- Infinity is not a real number, it is an idea.
- That is used to represent something without an end.
- Infinity cannot be measured.
- Infinity is greater than any real number.
- We use symbool ∞ to represent infinity.



Special properties of infinity

1. $\infty + \infty = \infty$
2. $-\infty + (-\infty) = -\infty$
3. $\infty \times \infty = \infty$
4. $-\infty \times -\infty = \infty$
5. $-\infty \times \infty = -\infty$
6. $x + \infty = \infty$
7. $-\infty + x = -\infty$
8. $x - (-\infty) = \infty$

Indeterminate form in Mathematics

- The term "indeterminate" is sometimes used as a synonym for **unknown** or **variable**.

- A mathematical expression can also be said to be indeterminate if it is not definitively or precisely determined.

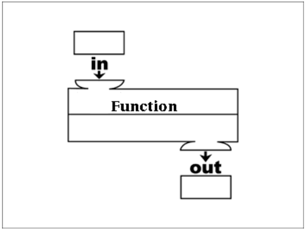
- There are seven indeterminate forms involving 0, 1, and ∞.

- They are,

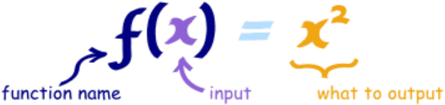
$$\frac{0}{0}, 0 \times \infty, \frac{\infty}{\infty}, \infty - \infty, 0^0, \infty^0, 1^\infty.$$

What is function?

- A function is something like a machine.
- It has an input and an output.
- The output is related somehow to the input.



- It is useful to give a function a name.
- $f(x)$ is the classic way of writing a function.
- The most common name is f, but you can have other names like $g, h, v, \ldots$ etc.



Example

Consider the function $f(x) = x^2 + 7x - 6$ and find followings.

(i) $f(2)$

(ii) $f(-2)$

(iii) $f(\sqrt{2})$

Introduction to Limits

Why do we need limits?

- In some situations, we cannot work something out directly.
- But we can see how it behaves as we get closer and closer.
- Let's consider below function as an example:

$$f(x) = \frac{(x^2 - 1)}{(x - 1)}$$

Why do we need limits?

Cont...

- Let's work it out for x=1:

$$\begin{aligned} f(1) &= \frac{(1^2-1)}{(1-1)} \\ &= \frac{0}{0} \end{aligned}$$

- We don't really know the value of 0/0.
- So we need another way of answering this.
- The limits can be used to give answer in such a situations.

Why do we need limits?
Cont...

Instead of trying to work it out for x=1, let's try approaching it closer and closer from $x < 1$:

x	$\dfrac{(x^2-1)}{(x-1)}$
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
...	...

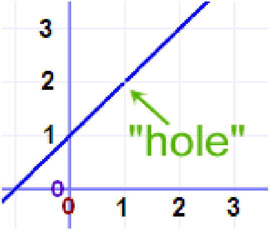
Why do we need limits?
Cont...

Instead of trying to work it out for x=1, let's try approaching it closer and closer from $x > 1$:

x	$\dfrac{(x^2-1)}{(x-1)}$
1.5	2.50000
1.1	2.10000
1.01	2.01000
1.001	2.00100
1.0001	2.00010
1.00001	2.00001
...	...

Why do we need limits?
Cont...

- Now we can see that as x gets close to 1, then $(x^2 - 1)/(x - 1)$ gets close to 2.
- When $x = 1$ we don't know the answer.
- But we can see that it is going to be 2.



Why do we need limits?
Cont...

- We want to give the answer "2" but can't, so instead mathematicians say exactly what is going on by using the special word "limit".

The limit of $(x^2 - 1)/(x - 1)$ as x approaches 1 is 2

- It can be written symbolicaly as:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Properties of limit

Suppose $f(x)$ and $g(x)$ are functions of x and a and c are constants. Then we have following properties for limit.

1. $\lim\limits_{x \to a} c = c.$
2. $\lim\limits_{x \to a} cx = c \lim\limits_{x \to a} x.$
3. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x).$
4. $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x).$
5. $\lim\limits_{x \to a} [f(x).g(x)] = \left(\lim\limits_{x \to a} f(x)\right) . \left(\lim\limits_{x \to a} g(x)\right).$
6. $\lim\limits_{x \to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$

Find the following limits:

(i) $\lim\limits_{x \to 3} 6.$

(ii) $\lim\limits_{x \to 2} (6x).$

(iii) $\lim\limits_{x \to -1} (x^3 + 7x^2 + 3x + 6).$

(iv) $\lim\limits_{t \to 3} (t^2 - 5t).$

(v) $\lim\limits_{x \to 2} (x^2 + 7\sqrt{x}).$

(vi) $\lim\limits_{x \to -5} (1/x).$

(vii) $\lim\limits_{x \to 4} (x + 5)(x + 6).$

(viii) $\lim\limits_{x \to 2} \dfrac{x^2 + 5x + 7}{x + 4}.$

(ix) $\lim\limits_{v \to 3} \left(\log(\sqrt{5} + 2)v^5\right).$

(x) $\lim\limits_{x \to 1} (x^2 + 5x + 2)(2x + 5).$

Example 1

Find the limit of $(4 - 3x)/(5 + x)$ as,

(i) $x \to 1$.

(ii) $x \to 4/3$.

(iii) $x \to -5$.

(iv) $x \to 0$.

Example 2

Find the following limits:

(i) $\lim_{x \to 0} \dfrac{x^3 - 7x^2 - 3x}{x}$.

(ii) $\lim_{x \to 1} \frac{x^2-1}{x-1}$.

(iii) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

(iv) $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a}$.

(v) $\lim_{t \to 1} \dfrac{2 - 2t^2}{t - 1}$.

(vi) $\lim_{t \to -1} \dfrac{2 - 2t^2}{t - 1}$.

Example 3

Find the following limits:

(i) $\lim_{x\to\infty}(5x)$.

(ii) $\lim_{x\to\infty}(3x+7)$.

(iii) $\lim_{x\to\infty}(2x-100)$.

(iv) $\lim_{x\to-\infty}(2x)$.

(v) $\lim_{x\to\infty}\left(\frac{2x}{x+1}\right)$.

(vi) $\lim_{x\to\infty}\left(\frac{7x}{2x-5}\right)$.

(vii) $\lim_{x\to\infty}\left(\frac{7x^2+5x+1}{4x^2+3x+5}\right)$.

Excercise

Show the followings:

(i) $\lim_{x\to -1}(5x+4) = -1.$

(ii) $\lim_{x\to 1}(5x^2+7x+3) = 15.$

(iii) $\lim_{x\to -1} \frac{x^2+5x+3}{2x} = \frac{1}{2}.$

(iv) $\lim_{x\to \infty} \frac{2x^2+5x+3}{x^2+6x+9} = 2.$

(v) $\lim_{x\to \infty} \left(\frac{2x^2+x}{3x^2+4}\right) = \frac{2}{3}.$

(vi) $\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = 1.$

(vii) $\lim_{x\to 3} \left(\frac{x^2-9}{x-3}\right) = 6.$

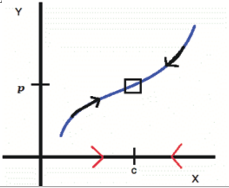
Right and left hand limits

If $f(x)$ approaches the value p as x approaches to c, we say p is the limit of the function $f(x)$ as x tends to c. That is

$$\lim_{x \to c} f(x) = p.$$

Then we can define right and left hand limit as follows:

$$\begin{aligned} \lim_{x \to c^-} f(x) &= p \Leftarrow \text{Left Hand Limit,} \\ \lim_{x \to c^+} f(x) &= p \Leftarrow \text{Right Hand Limit} \end{aligned}$$



Example 1

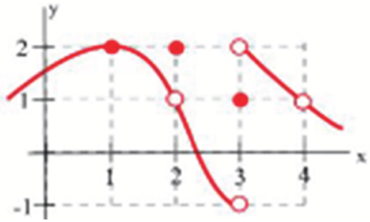
Use the graph to determine following limits:

(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 3} f(x)$

(d) $\lim_{x \to 4} f(x)$



Example 1
Solution

(a) $\lim_{x \to 1} f(x) = 2$

(b) $\lim_{x \to 2} f(x) = 1$

(c) $\lim_{x \to 3} f(x) \Leftarrow$ does not exist

(d) $\lim_{x \to 4} f(x) = 1$

Example 2

The function f is defined by:

$$f(x) = \begin{cases} x+3 & \text{if } x \leq 2 \\ -x+7 & \text{if } x > 2. \end{cases}$$

What is $\lim_{x \to 2} f(x)$.

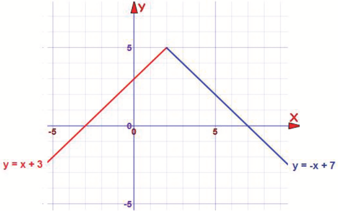
Example 2
Solution

Let's consider the left and right hand side limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x+3 = 5$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -x+7 = 5$$

We get same value for left and right hand limits. Hence

$$\lim_{x\to 2} f(x) = 5.$$



Thank You