Applied Statistics I
(IMT224β/AMT224β)

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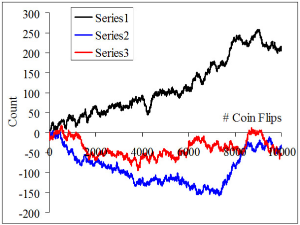
Statistical Applications With Probability Models

Introduction

A **probability model** is a mathematical representation of a random phenomenon. In here we mainly focus on following probability models.

1. Binormial model
2. Bernoulli model
3. Poisson model
4. Normal model



Binormial model

Binomial experiment

A **binomial experiment** (also known as a **Bernoulli trial**) is a statistical experiment that has the following properties:

- The experiment consists of n repeated trials.
- Each trial can result in just two possible outcomes. We call one of these outcomes a **success** and the other, a **failure**.
- The probability of success, denoted by p, is the same on every trial.
- The trials are independent; that is, the outcome on one trial does not affect the outcome on other trials.

Example for binomial experiment

Suppose you flip a coin 2 times and count the number of times the coin lands on heads. This is a binomial experiment because:

- The experiment consists of repeated trials. We flip a coin 2 times.
- Each trial can result in just two possible outcomes, heads or tails.
- The probability of success is constant (0.5) on every trial.
- The trials are independent; that is, getting heads on one trial does not affect whether we get heads on other trials.

Notation

The following notation is used, when we talk about binomial probability.

- k: The number of successes.
- n: The number of trials.
- p: The probability of success on an individual trial.
- q: The probability of failure on an individual trial.
- $\text{bin}(k; n, p)$: Binomial probability, the probability that an n-trial binomial experiment results in exactly k successes, when the probability of success on an individual trial is p.

Binomial distribution

- A **binomial random variable** is the number of successes k in n repeated trials of a binomial experiment.
- The probability distribution of a binomial random variable is called a **binomial distribution**.
- The **binomial probability** refers to the probability that a binomial experiment results in exactly k successes.

Binomial formula

Suppose a binomial experiment consists of n trials and results in k successes. If the probability of success on an individual trial is p, then the binomial probability is:

$$Pr(X = k) = \mathrm{bin}(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}; \qquad k = 0, 1, 2, ..., n.$$

Notation:

$$X \sim \mathrm{bin}(n, p),$$

where mean of X is np and variance of X is npq. That is,

$$\begin{aligned} E(X) &= np \\ Var(X) &= npq. \end{aligned}$$

Example 1

Suppose a die is tossed 5 times. What is the probability of getting outcome four exactly two times?

Solution

This is a binomial experiment in which the number of trials is equal to 5, the number of successes is equal to 2, and the probability of success on a single trial is $1/6$ or about 0.167. Therefore, the binomial probability is:

$$
\begin{aligned}
Pr(X=2)=\mathrm{bin}(2;5,0.167) &= \binom{5}{2}(0.167)^2(1-0.167)^{5-2} \\
&= \binom{5}{2}(0.167)^2(0.833)^3 \\
&= 0.161
\end{aligned}
$$

Example 2

The probability that a student is accepted to a prestigeous college is 0.3. If 5 students from the same school apply, what is the probability that at most 2 are accepted?

Solution

$$
\begin{aligned}
\mathrm{bin}(k \leq 2; 5, 0.3) &= \mathrm{bin}(k = 0; 5, 0.3) + \mathrm{bin}(k = 1; 5, 0.3) \\
&+ \mathrm{bin}(k = 2; 5, 0.3) \\
&= \binom{5}{0}(0.3)^0(1-0.3)^{5-0} \\
&+ \binom{5}{1}(0.3)^1(1-0.3)^{5-1} \\
&+ \binom{5}{2}(0.3)^2(1-0.3)^{5-2} \\
&= 0.8369
\end{aligned}
$$

Example 3

A manufacturer of metal pistons finds that on the average, 12% of his pistons are rejected because they are either oversize or undersize. What is the probability that a batch of 10 pistons will contain

(a) no more than 2 rejects?

(b) at least 2 rejects?

Solution

(a) Let X = number of rejected pistons

In this case, "success" means rejection!

Here, $n = 10, p = 0.12, q = 0.88$.

The probability of getting no more than 2 rejects is:

$$Pr(X \leq 2) = Pr(X = 0) + Pr(X = 1) + Pr(X = 2)$$

Cont...

$$
\begin{aligned}
Pr(X=0) &= \binom{10}{0}(0.12)^{0}(0.88)^{10-0} = 0.2785 \\
Pr(X=1) &= \binom{10}{1}(0.12)^{1}(0.88)^{10-1} = 0.37977 \\
Pr(X=2) &= \binom{10}{2}(0.12)^{2}(0.88)^{10-2} = 0.23304
\end{aligned}
$$

So, the probability of getting no more than 2 rejects is:

$$
\begin{aligned}
Pr(X \leq 2) &= 0.2785 + 0.37977 + 0.23304 \\
&= 0.89131
\end{aligned}
$$

Cont...

(b) We could work out all the cases for $X = 2, 3, 4, ..., 10$, but it is much easier to proceed as follows:

$$\begin{aligned}
\text{Probability of} \quad & \\
\text{at least 2 rejects} \quad &= 1 - Pr(X \leq 1) \\
&= 1 - (Pr(X = 0) + Pr(X = 1)) \\
&= 1 - (0.2785 + 0.37977) \\
&= 0.34173
\end{aligned}$$

Example 4

A certain type of missiles can attack a target with probability $p = 0.2$. Suppose we fire n missiles. If the probability of attacking the target at least by one of the missiles is at least 90%. Find n.

Solution

X=Number of missiles hit the target

$$
\begin{aligned}
Pr(X=0) &= \binom{n}{0}(0.2)^0(0.8)^n \\
&= (0.8)^n
\end{aligned}
$$

The probability of attacking the target by at least one missile is

$$
\begin{aligned}
&= 1-(0.8)^n \\
1-(0.8)^n &\geq 0.9 \\
0.1 &\geq (0.8)^n \\
\log(0.1) &\leq n\log(0.8) \\
n &\geq \frac{\log 0.1}{\log 0.8} \\
n &\geq 10.31 \\
n &= 11
\end{aligned}
$$

Excercise

(a) Bits are sent over a communications channel in packets of 12. If the probability of a bit being corrupted over this channel is 0.1 and such errors are independent, what is the probability that no more than 2 bits in a packet are corrupted?

(b) If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits?

(c) Let X denote the number of packets containing 3 or more corrupted bits. What is the probability that X will exceed its mean by more than 2 standard deviations?

Distribution of summation of binomial random variables

$$\begin{aligned}
\text{Let } X_i &\sim \text{bin}(n_i, p) \\
\text{Let } Y &= \sum_{i=1}^{k} X_i; \quad X_i\text{'s are independent} \\
\text{Then } Y &\sim \text{bin}\left(\sum_{i=1}^{k} n_i, p\right) \\
E(Y) &= \sum_{i=1}^{k} n_i p \\
Var(Y) &= \sum_{i=1}^{k} n_i pq
\end{aligned}$$

Example

Consider the two pair of coins. Let

X=Number of heads in 100 tosses of the first coin

Y=Number of heads in 50 tosses of the second coin

Find the probability that $X + Y \leq 140$.

Solution

$$
\begin{aligned}
X &\sim \text{bin}\left(100, \frac{1}{2}\right) \\
Y &\sim \text{bin}\left(50, \frac{1}{2}\right) \\
X + Y &\sim \text{bin}\left(150, \frac{1}{2}\right) \\
\Pr(X + Y \leq 140) &= \sum_{k=0}^{140} \binom{150}{k} \left(\frac{1}{2}\right)^{k} \left(1 - \frac{1}{2}\right)^{150-k} \quad \text{or} \\
\Pr(X + Y \leq 140) &= 1 - \Pr(X + Y > 140) \\
&= 1 - \sum_{k=141}^{150} \binom{150}{k} \left(\frac{1}{2}\right)^{k} \left(1 - \frac{1}{2}\right)^{150-k}
\end{aligned}
$$

Bernoulli model

Bernoulli experiment

- A single experiment which can have one of two possible outcomes is called **Bernoulli experiment**.
- The two possible outcomes are "success" and "failure".
- The Binomial experiment is an n times repeated Bernoulli trial.
- When a binomial trial occurs once we get a Bernoulli trial.

Examples for Bernoulli experiments

- Flipping a coin.
- Rolling a die, where a six is "success" and everything else a "failure".
- Either you pass an exam or you do not pass an exam.
- Either you get the job you applied for or you do not get the job.

Bernoulli distribution

For a Bernoulli random variable probability distribution is given by

$$Pr(X = k) = p^k (1-p)^{1-k} \quad k = 0, 1.$$

- Where $k = 0$ and $k = 1$ represent two outcomes.
- When $k = 0$ we get a failure with probability $1 - p = q$.
- When $k = 1$ we get a success with probability p.

Notation

If X is a random variable which has a Bernoulli distribution with success probability p, then it is denoted by

$$X \sim \mathrm{Ber}(p).$$

Then mean of X is p and variance of X is pq. That is,

$$\begin{aligned} E(X) &= p \\ Var(X) &= pq. \end{aligned}$$

Distribution of summation of Bernoulli random variables

For $X_i, i = 1, 2, ..., n$ Bernoulli random variables,

$$\begin{aligned} X_i &\sim \mathrm{Ber}(p) \\ \sum_{i=1}^{n} X_i &\sim \mathrm{bin}(n, p) \end{aligned}$$

Example

Suppose that in each week a person buys a lottery ticket which gives him a chance of $1/100$ of a win.

(a) What is the chance of no wins in a week?

(b) What is the chance of 3 wins in the year (approximately)?

Solution

(a)

$$
\begin{aligned}
X &= \text{number of wins in a week, X=0, 1} \\
X &\sim \mathrm{Ber}(0.01) \\
\Pr(X = 0) &= (0.01)^0(1 - 0.01)^1 \\
&= 0.99
\end{aligned}
$$

Cont...

(b)

$$
\begin{aligned}
X &\sim \mathrm{Ber}(0.01) \\
1 \text{ year} &\Rightarrow 52 \text{ weeks} \\
\sum_{i=1}^{52} X_i &\sim \mathrm{bin}(52, 0.01) \\
\text{Let } Y &= \sum_{i=1}^{n} X_i \\
\Pr(Y = 3) &= \binom{52}{3} (0.01)^3 (1 - 0.01)^{49} \\
&= 0.013
\end{aligned}
$$

Poisson model

Poisson experiment

- A Poisson experiment examines the number of times an event occurs during a specified interval. The interval could be anything-a unit of time, length, volume, etc.
- The number of successes in two disjoint time intervals is independent.

Examples for Poisson experiments

- Car accidents
- Number of typing errors on a page
- Failure of a machine in one month
- The number of bacteria on a plate

Poisson formula

The probability distribution of a Poisson random variable, X representing the number of successes occurring in a given time interval or a specified region of space is given by the formula:

$$Pr(X = k) = \frac{e^{-\mu}\mu^{k}}{k!}$$

where

$$\begin{aligned} k &= 0, 1, 2, 3... \\ e &= 2.71828 \\ \mu &= \text{mean number of successes in the given time} \\ &\quad\ \text{interval or region of space.} \end{aligned}$$

Mean and variance of Poisson distribution

- If μ is the average number of successes occurring in a given time interval or region in the Poisson distribution, then the mean and the variance of the Poisson distribution are both equal to μ.

$$
\begin{aligned}
X &\sim \mathrm{Pois}(\mu) \\
E(X) &= \mu \\
Var(X) &= \mu.
\end{aligned}
$$

- In a Poisson distribution, only one parameter, μ is needed to determine the probability of an event.

Example 1

Suppose web site of the Department of Mathematics is visited by 12 people daily. What is the probability, that the web site will have 20 visitors a day?

Solution

$$
\begin{aligned}
Pr(X = k) &= \frac{e^{-\mu}\mu^k}{k!} \\
k &= 20 \\
e &= 2.71828 \\
\mu &= 12 \\
Pr(X = 20) &= \frac{e^{-12}12^{20}}{20!} \\
&= 0.0097
\end{aligned}
$$

Example 2

A company makes electric motors. The probability an electric motor is defective is 0.01. What is the probability that a sample of 300 electric motors will contain exactly 5 defective motors?

Solution

The average number of defectives in 300 motors is $\mu = np = 0.01 \times 300 = 3$.

The probability of getting 5 defectives is:

$$
\begin{aligned}
Pr(X = k) &= \frac{e^{-\mu}\mu^k}{k!} \\
k &= 5 \\
e &= 2.71828 \\
\mu &= 3 \\
Pr(X = 5) &= \frac{e^{-3}3^5}{5!} \\
&= 0.10082
\end{aligned}
$$

Cont...

This problem looks similar to a binomial distribution problem, that we met in the last section.
If we do it using binomial, with $n = 300, k = 5, p = 0.01$ and $q = 0.99$, we get:

$$\begin{aligned}\Pr(X = 5) &= \binom{300}{5}(0.01)^5(0.99)^{295} \\ &= 0.10099\end{aligned}$$

We see that the result is very similar. We can use binomial distribution to approximate Poisson distribution (and vice-versa) under certain circumstances.

Example 3

If electricity power failures occur according to a Poisson distribution with an average of 3 failures every twenty weeks, calculate the probability that there will not be more than one failure during a particular week.

Solution

Not more than one failure means we need to include the probabilities for zero failures plus one failure.

$$\begin{aligned}
\Pr(X=0)+\Pr(X=1) &= \frac{e^{-0.15}(0.15)^0}{0!}+\frac{e^{-0.15}(0.15)^1}{1!} \\
&= 0.98981
\end{aligned}$$

Distribution of summation of Poisson random variables

$$\begin{aligned} \text{let } X_i &\sim \text{Pois}(\mu_i) \\ \text{Let } Y &= \sum_{i=1}^{n} X_i \; ; \quad X_i\text{'s are independent} \\ \text{Then } Y &\sim \text{Pois}\left(\sum_{i=1}^{n} \mu_i\right). \end{aligned}$$

Example 1

A book contains 300 pages. The number of mistakes on a page is distributed with a Poisson distribution with average 2.

(a) Find the probability that number of mistakes in the book is more than 630.

(b) Find the probability that there is no mistakes in the book.

(c) Find the expected number of pages with no mistakes.

Solution

(a)

$$
\begin{aligned}
X_i &= \text{number of mistakes on } i^{th} \text{ page} \\
X_i &= \text{Pois}(2) \\
\text{Let } Y &= \text{number of mistakes in the book} \\
Y &= \sum_{i=1}^{300} X_i \\
Y &\sim \text{Pois}\left(\sum_{i=1}^{300} \mu_i\right) \\
Y &\sim \text{Pois}(300 \times 2) \\
Y &\sim \text{Pois}(600) \\
\Pr(Y > 630) &= 1 - \Pr(Y \leq 630) = 1 - \sum_{k=0}^{630} \frac{e^{-600}(600)^k}{k!}
\end{aligned}
$$

Cont...

(b)

$$\Pr(Y=0)=\frac{e^{-600}600^{0}}{0!}=e^{-600}\simeq 0$$

(c)

$$\Pr(X=0)=\frac{e^{-2}2^{0}}{0!}=e^{-2}\simeq 0.135$$

The expected number of pages with no mistakes

$$\begin{aligned} &= np \\ &= 300\times 0.135 \\ &= 40.5 \\ &\simeq 41 \end{aligned}$$

Example 2

Let X and Y be two independent Poisson random variables with parameters 1 and 2 respectively.

(a) Find the probability that $X + Y > 2$.

(b) Find the probability that $X = 1$ and $Y = 2$.

Solution

(a)

$$
\begin{aligned}
X &\sim \mathrm{Poiss}(1) \\
Y &\sim \mathrm{Poiss}(2) \\
X+Y &\sim \mathrm{Poiss}(3) \\
\Pr(X+Y>2) &= 1-\Pr(X+Y\leq 2) \\
&= 1-\Pr(X+Y=0)-\Pr(X+Y=1)- \\
&\quad \Pr(X+Y=2) \\
&= 1-\frac{e^{-3}3^0}{0!}-\frac{e^{-3}3^1}{1!}-\frac{e^{-3}3^2}{2!} \\
&= 1-e^{-3}\left(1+3+\frac{3^2}{2!}\right)
\end{aligned}
$$

Solution

(b)

$$\begin{aligned} \Pr(X = 1 \text{ and } Y = 2) &= \Pr(X = 1)\, .\, \Pr(Y = 2) \\ &= \frac{e^{-1}1^1}{1!} \cdot \frac{e^{-2}2^2}{2!} \\ &= 0.0993 \end{aligned}$$

Exercise 1

An average of 0.1 customers per minutes will arrive at a check point. Find the probability that less than two customers arrives at the check point within 4 consecutive minutes.

Exercise 2

A bank receives an average of 6 bad checks per day. Find the probability that it will receive 4 bad checks per day.

Normal model

Continuous random variables

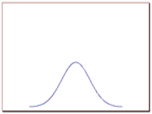
- A continuous random variable is one which takes an infinite number of possible values.
- **Eg:** height, weight, the amount of sugar in an orange.
- A continuous random variable is not defined at specific values.
- It is defined over an interval of values.
- The probability of observing any single value is equal to 0, since the number of values which may be assumed by the random variable is infinite.

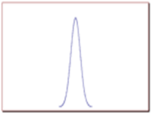
Normal distribution

- The **normal** (or **Gaussian**) distribution is a continuous probability distribution.
- The graph of the normal distribution depends on two factors, the mean and the standard deviation.
- The mean of the distribution determines the location of the center of the graph, and the standard deviation determines the height and width of the graph.

Cont...

- When the standard deviation is large, the curve is short and wide.
- When the standard deviation is small, the curve is tall and narrow.
- All normal distributions look like a symmetric, bell-shaped curve.





Probability distribution

The random variable X is said to be normally distributed if its probability density function $f_X(x)$ is

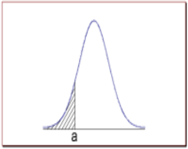
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2};$$

where μ and σ^2 are real and $\sigma > 0$.

Probability and the normal curve

The normal distribution is a continuous probability distribution. This has several implications for probability.

- The total area under the normal curve is equal to 1.
- The probability that a normal random variable X equals any particular value is 0.
- The probability that X is greater than "a" equals the area under the normal curve bounded by "a" and plus infinity.
- The probability that X is less than "a" equals the area under the normal curve bounded by "a" and minus infinity.



Notation

If X is a normally distributed random variable with mean μ and standard deviation σ, then it is denoted by,

$$\begin{aligned} X &\sim N(\mu, \sigma^2) \text{ and where} \\ E(X) &= \mu \\ Var(X) &= \sigma^2 \end{aligned}$$

Standard normal distribution

- The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1.
- Normal distributions can be transformed to standard normal distributions by the formula:

$$z = \frac{X - \mu}{\sigma}.$$

- The standard normal distribution is sometimes called the z distribution.

$$\begin{aligned} E(z) &= 0 \\ Var(z) &= 1 \end{aligned}$$

Why do we need standard normal distribution?

- Normally, you would work out the probablity for continuous random variable by doing some integration.
- However, it is impossible to do this for the normal distribution and so results have to be looked up in statistical tables.
- The mean and variance of the normal distribution can be any value and so clearly there can't be a statistical table for each one.
- Therefore, we convert any normal distribution to the standard normal distribution.

Example 1

Find the following probabilities:

(a) $P(Z > 1.06)$.

(b) $P(Z < -2.15)$.

(c) $P(1.06 < Z < 4.00)$.

(d) $P(-1.06 < Z < 4.00)$.

Example 2

It was found that the mean length of 100 parts produced by a lathe was 20.05 mm with a standard deviation of 0.02 mm. Find the probability that a part selected at random would have a length

(a) between 20.03 mm and 20.08 mm.

(b) between 20.06 mm and 20.07 mm.

(c) less than 20.01 mm.

(d) greater than 20.09 mm.

Example 3

(a) Suppose that the amount of cosmic radiation to which a person is expected while flying by a jet is random variable having a normal distribution with mean $=4.35$ (units) and standard deviation $=0.59$ (units). What is the probability that a person will be expected to have more than 5 (units) of cosmic radiation on such a flight.

(b) If the number of travelers in a flight is 100, what is the expected number of travelers exposed to cosmic radiation more than 5 (units).

Solution

X=amount of cosmic radiation on a flight

$$
\begin{aligned}
X &\sim N(4.35, 0.59^2) \\
Pr(X > 5) &= Pr\left(\frac{X-\mu}{\sigma} > \frac{5-\mu}{\sigma}\right) \\
&= Pr\left(\frac{X-4.35}{0.59} > \frac{5.00-4.35}{0.59}\right) \\
&= Pr(z > 1.102) \\
&= 1 - 0.8643 \\
&= 0.1357
\end{aligned}
$$

Expected number of travelers exposed to cosmic radiation more than 5 (units) $= 100 \times 0.1357$.

Example 4

In school term test, a candidate fails if he obtains less than 40 marks out of the 100 marks and he has to obtain at least 76 marks in order to pass examination with a distinction. The top 10% of the students have received passes with distinction and the bottom 30% of the students have failed the examination. Assuming that the distribution of marks is normal, find the mean and the standard deviation of the marks.

Solution

Let X be the marks of any student
Let

$$\begin{aligned}
X &\sim N(\mu, \sigma^2) \\
Pr(X < 40) &= 30\% = 0.3 \\
Pr\left(\frac{X-\mu}{\sigma} < \frac{40-\mu}{\sigma}\right) &= 0.3 \\
Pr\left(z < \frac{40-\mu}{\sigma}\right) &= 0.3 \\
Pr(z < -0.52) &= 0.3 \\
\frac{40-\mu}{\sigma} &= -0.52 \qquad (1)
\end{aligned}$$

Solution

$$\begin{aligned}
Pr(x \geq 76) &= 10\% = 0.1 \\
Pr\left(\frac{X-\mu}{\sigma} \geq \frac{76-\mu}{\sigma}\right) &= 0.1 \\
\frac{76-\mu}{\sigma} &= 1.28 \qquad (2) \\
\text{From (1) and (2)} & \\
\frac{40-\mu}{76-\mu} &= -\frac{0.52}{1.28} \\
\mu &= 50.4 \\
\sigma &= 20
\end{aligned}$$

Distribution of summation of normal random variables

$$\begin{aligned} \text{Let } X_i &\sim N(\mu_i, \sigma_i^2);\ X_i'\text{s are independent.} \\ \text{Let } Y &= \sum_{i=1}^{n} X_i \\ \text{Then } \ Y &\sim \left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right). \end{aligned}$$

Cont...(For common μ and σ)

$$
\begin{aligned}
\text{Let } X_i &\sim N(\mu, \sigma^2);\ X_i'\text{s are independent.} \\
\text{Let } Y &= \sum_{i=1}^{n} X_i. \\
\text{Then } \ Y &\sim \left(n\mu, n\sigma^2\right).
\end{aligned}
$$

Distribution of $\overline{X}$ for common μ and σ

$$
\begin{aligned}
\text{Let } X_i &\sim N(\mu, \sigma^2);\ X_i'\text{s are independent.} \\
\text{Let } \overline{X} &= \frac{\sum_{i=1}^{n} X_i}{n};\ \text{for independent } X_i. \\
\text{Then } \overline{X} &\sim N\left(\mu, \frac{\sigma^2}{n}\right).
\end{aligned}
$$

Distribution of difference of two normal random variables

If X_1 and X_2 are independent normally distributed random variables with distribution,

$$\begin{aligned} X_1 &\sim N(\mu_1, \sigma_1^2) \\ X_2 &\sim N(\mu_2, \sigma_2^2), \\ \text{Then } X_1 - X_2 &\sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2). \end{aligned}$$

Example 1

A foot bridge of negligible weight is capable of carrying out a maximum weight of $400Kg$ at once. If 9 persons go by it at the same time, find the probability that the foot bridge is out of dangerous. You may assume that the weight of a person has a normal distribution with the mean of $50Kg$ and the standard deviation of $6Kg$.

Solution

$$\begin{aligned}
\text{Let } X &= \text{weight of a person} \\
X &\sim N(50, 6^2) \\
\text{Let } Y &= \sum_{i=1}^{9} X_i \\
\text{Then } Y &\sim N(9 \times 50, 9 \times 6^2) \\
Y &\sim N(450, 9 \times 6^2)
\end{aligned}$$

Cont...

$$
\begin{aligned}
Pr(Y < 400) &= Pr\left(\frac{Y-450}{\sqrt{9\times 6^2}} < \frac{400-450}{\sqrt{9\times 6^2}}\right) \\
&= Pr\left(z < -\frac{50}{18}\right) \\
&= Pr(z < -2.778) \\
&= 1 - 0.9973 \\
&= 0.0027
\end{aligned}
$$

Example 2

The nicotine content of a certain brand of cigarettes is normally distributed with mean equal to $25mg$ and standard deviation equal to $4mg$. A random sample of 25 cigarettes is selected. What is the probability that the average nicotine content in the sample is at least $24mg$.

Solution

$$
\begin{aligned}
\text{Let } X &= \text{the nicotine content of the cigarette} \\
X &\sim N(25, 4^2) \\
\text{Let } \overline{X} &\sim \text{average/mean of nicotine in the sample} \\
\overline{X} &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \\
\mu &= 25 \\
\frac{\sigma^2}{n} &= \frac{4^2}{25} = \frac{16}{25} \\
\overline{X} &\sim N\left(25, \frac{16}{25}\right)
\end{aligned}
$$

Cont...

$$
\begin{aligned}
Pr(\overline{X} \geq 24) &= Pr\left(\frac{\overline{X} - 25}{4/5} > \frac{24 - 25}{4/5}\right) \\
&= Pr\left(z > -\frac{5}{4}\right) \\
&= Pr(z > -1.25) \\
&= 0.8944
\end{aligned}
$$

Example 3

Let X_1 and X_2 represent the life time of two types of bulbs A and B.

$$X_1 \sim N(1010, 5^2) \quad \text{and } X_2 \sim N(1020, 10^2)$$

What is the probability that A has higher life time that of B.

Solution

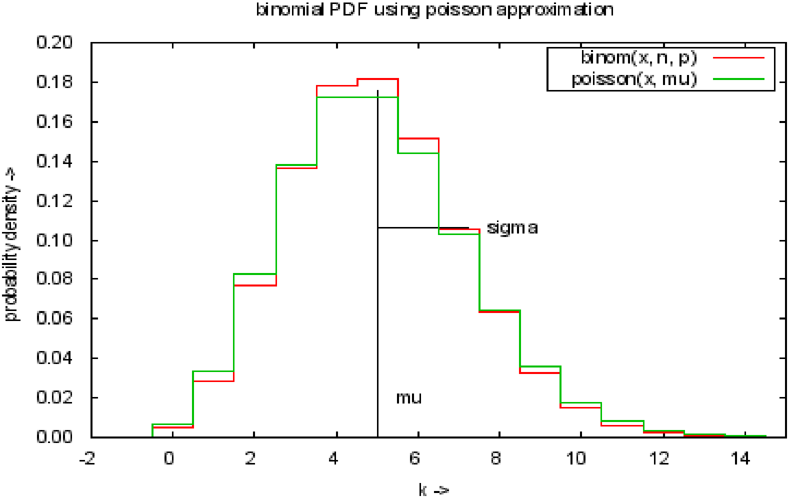
$$
\begin{aligned}
Pr(X_1 > X_2) &= Pr(X_1 - X_2 > 0) \\
X_1 - X_2 &\sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2) \\
\text{Since } X_1 &\sim N(1010, 5^2) \quad \text{ and } X_2 \sim N(1020, 10^2) \\
X_1 - X_2 &\sim N(-10, 125) \\
Pr(X_1 - X_2 > 0) &= Pr\left(\frac{(X_1 - X_2) - (-10)}{\sqrt{125}} > \frac{0 - (-10)}{\sqrt{125}}\right) \\
&= Pr\left(z > \frac{10}{\sqrt{125}}\right) \\
&= Pr(z > 0.89) \\
&= 1 - 0.8133 \\
&= 0.1867
\end{aligned}
$$

Probability approximations

The Poisson approximation to Binomial probabilities

- The binomial distribution converges towards the Poisson distribution as the number of trials goes to infinity while the product np remains fixed.
- Therefore the Poisson distribution with parameter $\mu = np$ can be used as an approximation to $\text{bin}(n, p)$ of the binomial distribution if n is sufficiently large and p is sufficiently small.
- This approximation is good if $n \geq 20$ and $p \leq 0.05$, or if $n \geq 100$ and $np \leq 10$.

Poisson approximation when $n = 50; p = 0.1$



Example

A certain disease occurs in 1.2% of the population. 100 people are selected at random from the population.

(a) Find the probability that no one has the disease.

(b) Find the probability that exactly 2 have the disease.

Solution

- This is a binomial experiment.
- The number of trials $n = 100$.
- $p = 0.012$.
- This satisfies $n \geq 20$ and $p \leq 0.05$.
- Therefore we can approximate binomial probabilities using Poisson distribution.
- For this approximation Poisson parameter, $\mu = np = 100 \times 0.012 = 1.2$.

Cont...

(a)

$$Pr(X=k) = \frac{e^{-\mu}\mu^k}{k!}$$
$$Pr(X=0) = \frac{e^{-1.2}(1.2)^0}{0!} \simeq 0.301$$

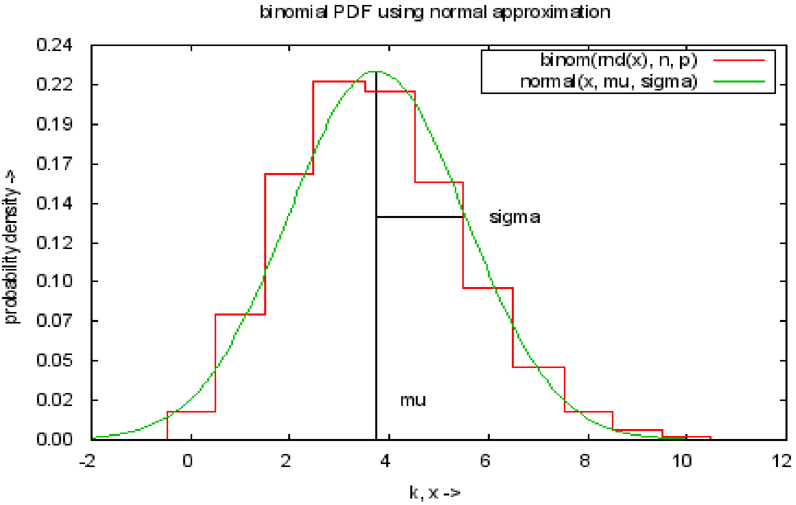
(b)

$$Pr(X=2) = \frac{e^{-1.2}(1.2)^2}{2!} \simeq 0.217$$

Normal approximation to the Binomial distribution

- A discrete random variable can take on only specified values while a continuous random variable can take on any values within an interval around those specified values.
- Hence, when using the normal distribution to approximate the binomial distributions, a correction for continuity adjustment is needed.

Normal approximation when $n = 25; p = 0.15$



Cont...

- A continuous distribution (such as the normal), the probability of obtaining a particular value of a random variable is zero.
- When the normal distribution is used to approximate a discrete distribution, a correction for continuity adjustment can be employed so that the probability of a specific value of the discrete distribution can be approximated.

Cont...

- The binomial distribution is symmetric (like the normal distribution) whenever $p = 0.5$.
- When $p \neq 0.5$ the binomial distribution will not be symmetric.
- However, the closer p is to 0.5 and the larger the number of sample observations n, the more symmetric the distribution becomes.

Cont...

- The larger the number of observations in the sample, it is difficult to compute the exact probabilities of success by use of binomial formula.
- Fortunately, though, whenever the sample size is large, the normal distribution can be used to approximate the exact probabilities of success.

Condition for normal approximation

- The normal distribution provides a good approximation to the binomial when n is large and p is close to 0.5.
- However when n is not fairly large and p differs from $1/2$ (but p is not close to zero or one) the normal approximation can be used when both np and $n(1-p)$ are greater than 5.

Normal approximation for exact number of success

The binomial probability $Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ can be approximated by the normal probability $Pr(k - 0.5 \leq X \leq k + 0.5)$ considering the continuity correction.

Normal approximation for one sided binomial probabilities

- $Pr(X \leq k)_{\text{bin}} \simeq Pr(X \leq k + 0.5)_{\text{normal}}$
- $Pr(X \geq k)_{\text{bin}} \simeq Pr(X \geq k - 0.5)_{\text{normal}}$
- $Pr(X < k)_{\text{bin}} = Pr(X \leq k - 1)_{\text{bin}} \simeq Pr(X \leq k - 0.5)_{\text{normal}}$
- $Pr(X > k)_{\text{bin}} = Pr(X \geq k + 1)_{\text{bin}} \simeq Pr(X \geq k + 0.5)_{\text{normal}}$

Normal approximation for two sided binomial probabilities

- $Pr(a < X < b)_{\text{bin}} \simeq Pr(a + 0.5 \leq X \leq b - 0.5)_{\text{normal}}$
- $Pr(a \leq X \leq b)_{\text{bin}} \simeq Pr(a - 0.5 \leq X \leq b + 0.5)_{\text{normal}}$

Example 1

In a production process 10% units produced are defective. Let X be the number of defectives found in a sample of 100 units selected at random.

(a) Find $Pr(X \leq 13)$.

(b) Find $Pr(X > 7)$.

(c) Find $Pr(5 \leq X < 11)$.

Solution

(a)

$$\begin{aligned} X &- \text{number of defectives} \\ X &\sim \text{bin}(100, 0.1) \\ Pr(X \leq 13) &= \sum_{k=0}^{13} \binom{100}{k} (0.1)^k (0.9)^{100-k} \end{aligned}$$

It is very hard to find the value of the right hand expression. But in here,

$$\begin{aligned} np &= 100 \times 0.1 = 10 > 5 \\ n(1-p) &= 90 > 5. \end{aligned}$$

Since np and $n(1-p)$ both greater than 5, we can use normal approximation to the binomial distribution.

Cont...

$$
\begin{aligned}
Pr(X \leq 13)_{\text{bin}} &\simeq Pr(X \leq 13.5)_{\text{normal}} \\
\mu &= np = 10 \\
\sigma &= \sqrt{npq} \\
&= 3 \\
Pr(X \leq 13.5)_{\text{normal}} &= Pr\left(\frac{\overline{x} - 10}{3} \leq \frac{13.5 - 10}{3}\right) \\
&= Pr\left(z \leq \frac{3.5}{3}\right) \\
&= Pr(z \leq 1.167) \\
&= 0.8790
\end{aligned}
$$

Cont...

(b)

$$
\begin{aligned}
Pr(X > 7)_{\text{bin}} = Pr(X \geq 8)_{\text{bin}} &= Pr(X \geq 7.5)_{\text{normal}} \\
&= Pr\left(\frac{\overline{x} - 10}{3} \geq \frac{7.5 - 10}{3}\right) \\
&= Pr(z \geq -0.833) \\
&= 0.7967
\end{aligned}
$$

Cont...

(c)

$$Pr(5 \leq X < 11) = \sum_{k=5}^{10} \binom{100}{k} (0.1)^k (0.9)^{100-k}$$

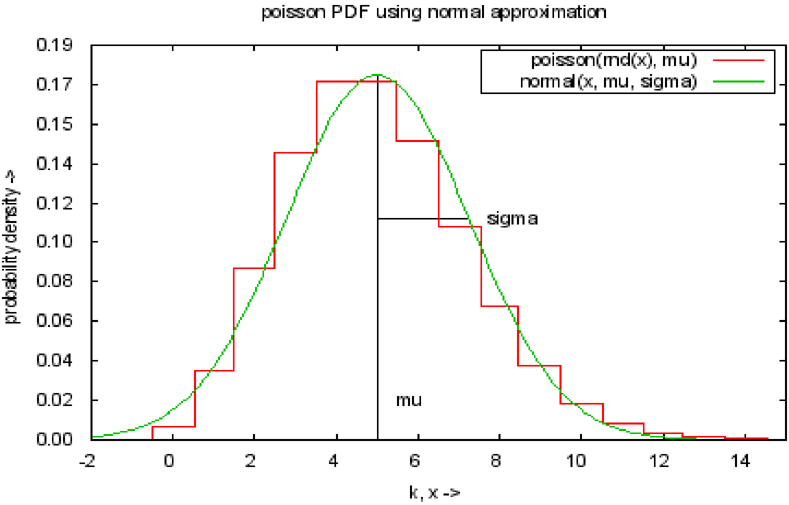
Using the normal approximation to the binomial distribution

$$\begin{aligned}
Pr(5 \leq X < 11)_{\text{bin}} &= Pr(5 \leq X \leq 10)_{\text{bin}} \\
&= Pr(4.5 \leq X \leq 10.5)_{\text{normal}} \\
&= Pr\left(\frac{4.5-10}{3} \leq \frac{\overline{x}-10}{3} \leq \frac{10.5-10}{3}\right) \\
&= Pr(-1.83 \leq z \leq 0.167) \\
&= 0.5675 - (1 - 0.9664) \\
&= 0.5339
\end{aligned}$$

Normal approximation to Poisson distribution

- The normal distribution can also be used to approximate the Poisson distribution whenever the parameter μ, the expected number of successes, equals or exceeds 5.

Normal approximation when $\mu = 5$



Cont...

- Since the value of the mean and the variance of a Poisson distribution are the same,

$$\begin{aligned} \sigma^2 &= \mu \\ \sigma &= \sqrt{\mu}. \end{aligned}$$

Cont...

Substituting into the transformation Equation,

$$
\begin{aligned}
z &= \frac{X-\mu}{\sigma} \\
z &= \frac{X-\mu}{\sqrt{\mu}}.
\end{aligned}
$$

So that, for large enough μ, the random variable z is approximately normally distributed.

Example

Suppose that at a certain automobile plant the average number of work stoppages per day due to equipment problems during the production process is 12.0. What is the approximate probability of having 15 or fewer work stoppages due to equipment problems on any given day?

Solution

In here $\mu = 12$ and it satisfies the condition that $\mu \geq 5$. So we can approximate this Poisson probability using the normal distribution.

$$\begin{aligned} Pr(X \leq 15)_{\text{Pois}} &\simeq Pr(X \leq 15.5)_{\text{normal}} \\ &= Pr\left(\frac{X-\mu}{\sqrt{\mu}} \leq \frac{15.5-\mu}{\sqrt{\mu}}\right)_{\text{normal}} \\ &= Pr\left(z \leq \frac{15.5-12}{\sqrt{12}}\right) \\ &= Pr(z \leq 1.01) \\ &= 0.8438 \end{aligned}$$

Cont...

Therefore, the approximate probability of having 15 or fewer work stoppages due to equipment problems on any given day is 0.8438. This approximation compares quite favorably to the exact Poisson probability, 0.8445.

Thanks