Applied Statistics I
(IMT224β/AMT224β)

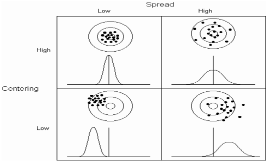
Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Measures of variation

Introduction

- As mentioned in previous chapter, we can characterize any set of data by measuring its central tendency, variation, and shape.
- In this chapter we are going to discuss about variation and some commonly used measures of variation.
- Measures of variation determine the range of the distribution, relative to the measures of central tendency.



Commonly used measures of variation

Some commonly used measures of variation are:

1. Range
2. Mean deviation
3. Variance/Standard deviation
4. Inter Quartile Range (IQR)
5. Semi Inter Quartile Range
6. Coefficient of Quartile Deviation
7. Five number summary

[1] Range

- Range is defined as the difference between the maximum and the minimum observation of the given data.
- If x_m denotes the maximum observation x_0 denotes the minimum observation then the range is defined as

$$x_m - x_0.$$

- The range is based on the two extreme observations.

[1] Range
Cont...

- It gives no weight to the central values of the data.
- It is a poor measure of dispersion.
- It does not give a good picture of the overall spread of the observations with respect to the center of the observations.

Example

Find the range of the following three group.

Group A: 30, 40, 40, 40, 40, 40, 50
Group B: 30, 30, 30, 40, 50, 50, 50
Group C: 30, 35, 40, 40, 40, 45, 50

Example
Solution

- In all the three groups the range is 50 -30 $=$ 20.
- In group A there is concentration of observations in the center.
- In group B the observations are friendly with the extreme corners.
- In group C the observations are almost equally distributed in the interval from 30 to 50.
- The range fails to explain these differences in the three groups of data.

[2] Mean deviation

Mean deviation about mean

- The mean deviation is defined as the mean of the absolute deviations of observations from the arithmetic mean.
- Let $x_1, x_2, ..., x_n$ denote n observations. The mean deviation about mean is defined as,

$$\frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{n} = \frac{\sum_{i=1}^{n} |d_i|}{n}$$

where,
$\overline{x}$=mean of the data
d_i=deviation of i^{th} observation from the mean.

Example

Calculate the mean deviation about mean from marks of nine students given below.

7, 4, 10, 9, 15, 12, 7, 9, 7

Example
Solution

$$
\begin{aligned}
\overline{x} &= \frac{7+4+10+9+15+12+7+9+7}{9} \\
&= \frac{80}{9} \\
&= 8.89 \\
\mathrm{M.D} &= \frac{|4-8.89|+|7-8.89|+...+|15-8.89|}{9} \\
&= \frac{21.11}{9} \\
&= 2.35
\end{aligned}
$$

Mean deviation about mean for data without class intervals

For a summarized data set with values the mean deviation about the mean is:

$$= \frac{\sum_{i=1}^{k} f_i \left| d_i \right|}{n},$$

$$\begin{aligned} n = \sum_{i=1}^{k} f_i \quad &- \quad \text{total number of observations} \\ k \quad &- \quad \text{number of different values} \\ d_i \quad &- \quad \text{deviation from the mean.} \end{aligned}$$

Example

Calculate the mean deviation of the following summarized data set.

x_i	f_i
2	1
4	4
6	6
8	4
10	1

Example
Solutoin

$$
\begin{aligned}
\text{mean} &= \frac{\sum_{i=1}^{5} f_i x_i}{\sum_{i=1}^{5} f_i} \\
&= \frac{2+16+36+32+10}{16} \\
&= \frac{96}{16} \\
&= 6
\end{aligned}
$$

Example
Solutoin⇒Cont...

x_i	f_i	d_i	$\lvert d_i \rvert$	$f_i \lvert d_i \rvert$
2	1	-4	4	4
4	4	-2	2	8
6	6	0	0	0
8	4	2	2	8
10	1	4	4	4

$$\text{The mean deviation about the mean} = \frac{24}{16} = 1.5$$

Mean deviation about mean for data with class intervals

For a summarized data set with class intervals the mean deviation about the mean is:

$$= \frac{\sum_{i=1}^{k} f_i \left| m_i - \overline{x} \right|}{n},$$

$$n = \sum_{i=1}^{k} f_i \quad - \quad \text{total number of observations}$$

$$m_i \quad - \quad \text{mid value of } i^{th} \text{ class}$$

$$k \quad - \quad \text{number of classes}$$

$$f_i \quad - \quad \text{frequency of } i^{th} \text{ class}$$

$$\overline{x} \quad - \quad \text{mean.}$$

Example

Calculate the mean deviation about mean from the following data.

Size of items	frequency
3-4	3
4-5	7
5-6	22
6-7	60
7-8	85
8-9	32
9-10	8

Example
Solution

Size of items	f_i	m_i	$f_i m_i$	$\lvert m_i - \overline{x} \rvert$	$f_i \lvert m_i - \overline{x} \rvert$
3-4	3	3.5	10.5	3.59	10.77
4-5	7	4.5	31.5	2.59	18.13
5-6	22	5.5	121.0	1.59	34.98
6-7	60	6.5	390.0	0.59	35.40
7-8	85	7.5	637.5	0.41	34.85
8-9	32	8.5	272.0	1.41	45.12
9-10	8	9.5	76.0	2.41	19.28
Total	217		1538.5		198.5

$$\text{mean} = \frac{\sum f_i m_i}{\sum f_i} = \frac{1538.5}{217} = 7.09$$

Example

Solution⇒Cont...

$$
\begin{aligned}
\mathrm{M.D} &= \frac{\sum_{i=1}^{7} f_i \left| m_i - \overline{x} \right|}{n} \\
&= \frac{198.53}{217} \\
&= 0.915
\end{aligned}
$$

[3] Variance

- Variance is another absolute measure of dispersion.
- It is defined as the average of the squared difference between each of the observations in a set of data and the mean.
- For a sample data the variance is denoted by S^2 and the population variance is denoted by σ^2.

Let $x_1, x_2, ..., x_n$ denote n observation of the sample and let $\overline{x}$ denote the sample mean. Then the sample variance S^2 is given by,

$$\begin{aligned} S^2 &= \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1} \\ S^2 &= \frac{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}{n-1}. \end{aligned}$$

Note: The positive square root of sample variance is called as sample standard deviation and it is denoted by S.

Let $x_1, x_2, ..., x_N$ denote N observation of the population and let μ denote the population mean. Then the population variance σ^2 is given by,

$$\sigma^2 = \frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}.$$

Note: The positive square root of population variance is called as population standard deviation and it is denoted by σ.

Example 1

Calculate the variance for the following sample data: 2, 4, 8, 6, 10, and 12.

Example 1
Solution

$$
\begin{aligned}
\overline{x} &= \frac{\sum_{i=1}^{6} x_i}{6} = \frac{42}{6} = 7 \\
S^2 &= \frac{\sum_{i=1}^{6}(x_i - 7)^2}{6-1} \\
&= \frac{(2-7)^2 + (4-7)^2 + \dots + (12-7)^2}{5} \\
&= \frac{70}{5} \\
&= 14
\end{aligned}
$$

Example 2

Consider the following distribution of data:

10, 18, 18, 12, 11, 15, 14

Calculate variance of above data.

Example 2
Solution

Population variance σ^2 is equal to

$$\begin{aligned}
\sigma^2 &= \frac{\sum_{i=1}^{N}(x_i-\mu)^2}{N} \text{ where } N = 7 \text{ and} \\
\mu &= \frac{10+18+12+11+15+14+13}{7} \\
&= 13.29 \\
\sigma^2 &= \frac{(10-13.29)^2+(18-13.29)^2+...+(14-13.29)^2}{7} \\
&= 6.204 \\
\sigma &= 2.491
\end{aligned}$$

Sample variance for summarized data without class intervals

The sample variance for summarized data with values is given by

$$S^2 = \sum_{i=1}^{k} \frac{f_i(x_i - \overline{x})^2}{n-1};$$

$$\begin{aligned} k &\quad - \quad \text{number of values} \\ n = \sum_{i=1}^{k} f_i &\quad - \quad \text{number of observations} \end{aligned}$$

Note: S=sample standard deviation.

Population variance for summarized data without class intervals

The population variance for summarized data with values is given by

$$\sigma^2 = \sum_{i=1}^{k} \frac{f_i(x_i - \mu)^2}{N};$$

$$\begin{aligned} k &\quad - \quad \text{number of values} \\ N = \sum_{i=1}^{k} f_i &\quad - \quad \text{population size} \end{aligned}$$

Note: σ=population standard deviation.

Example 1

Consider the number of children in 128 families which is summarized as follows.

x_i	f_i
0	20
1	15
2	25
3	30
4	18
5	10
6	6
7	3
8	1

Find the sample variance.

Example 1
Solution

x_i	f_i	$f_i x_i$	$(x_i - \overline{x})$	$(x_i - \overline{x})^2$	$f_i(x_i - \overline{x})^2$
0	20	0	-2.7	7.29	145.8
1	15	15	-1.7	2.89	43.35
2	25	50	-0.7	0.49	12.25
3	30	90	0.3	0.09	2.7
4	18	72	1.3	1.69	30.42
5	10	50	2.3	5.29	52.9
6	6	36	3.3	10.89	65.34
7	3	21	4.3	18.49	55.47
8	1	8	5.3	28.09	28.09
		342			436.321

Example 1
Solution⇒Cont...

$$
\begin{aligned}
\overline{x} &= \frac{\sum_{i=1}^{9} f_i x_i}{\sum_{i=1}^{9} f_i} \\
&= \frac{342}{128} \\
&= 2.67 \\
&\simeq 2.7 \\
S^2 &= \frac{\sum_{i=1}^{9} f_i (x_i - \overline{x})^2}{n-1} \\
&= \frac{436.32}{128-1} \\
&= 3.435
\end{aligned}
$$

Example 2

Consider the following summarized distribution of data and find the variance.

x_i	f_i
2	3
3	2
9	4
12	9
15	1
19	1

Example 2
Solution

x_i	f_i	$f_i x_i$	$(x_i - \mu)$	$(x_i - \mu)^2$	$f_i(x_i - \mu)^2$
2	3	6	-7.5	56.25	168.75
3	2	6	-6.5	42.25	84.30
9	4	36	-0.5	0.25	1.00
12	9	108	2.5	6.25	56.25
15	1	15	5.5	30.25	30.25
19	1	19	9.5	90.25	90.25
	20				431

Example 2
Solution⇒Cont...

$$\mu = \frac{\sum_{i=1}^{6} f_i x_i}{\sum_{i=1}^{6} f_i} = \frac{190}{20} = 9.5$$

$$\sigma^2 = \frac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{N}$$

$$= \frac{431}{20}$$

$$= 21.55$$

For a summarized sample data with class intervals
First formula

$$\text{Sample variance} = S^2 = \frac{\sum_{i=1}^{k} f_i(m_i - \overline{x})^2}{n-1}$$

$$\begin{aligned} k &\;-\; \text{number of class} \\ m_i &\;-\; \text{mid value of } i^{th} \text{ class} \\ \overline{x} &\;-\; \text{mean} \\ n &= \sum_{i=1}^{k} f_i \end{aligned}$$

For a summarized sample data with class intervals
Second formula

$$S^2 = \left[\frac{\sum_{i=1}^{k} f_i d_i^2 - \frac{\left(\sum_{i=1}^{k} f_i d_i\right)^2}{n}}{(n-1)} \right] w^2$$

w – class width of the class contains the assumed mean

f_i – frequency of i^{th} class

d_i – deviation of i^{th} class from the class that contains assumed mean.

Note: S=sample standard deviation

For a summarized population with class intervals

First formula

$$\text{Population variance} = \sigma^2 = \frac{\sum_{i=1}^{k} f_i (m_i - \mu)^2}{N}$$

$$\begin{aligned} k &- \text{number of class} \\ m_i &- \text{mid value of } i^{th} \text{ class} \\ \mu &- \text{population mean} \\ n &= \sum_{i=1}^{k} f_i \end{aligned}$$

For a summarized population with class intervals
Second formula

$$\sigma^2 = \left[\frac{\sum_{i=1}^{k} f_i d_i^2 - \frac{\left(\sum_{i=1}^{k} f_i d_i\right)^2}{N}}{N} \right] w^2$$

w – class width of the class contains the assumed mean

f_i – frequency of i^{th} class

d_i – deviation of i^{th} class from the class that contains assumed mean.

Note: σ=population standard deviation

Example

The following table illustrates the pocket money given to a sample of students on a particular day of school. Calculate the variance.

Money	f_i
12.5− <17.5	2
17.5− <22.5	22
22.5− <27.5	19
27.5− <32.5	14
32.5− <37.5	13
37.5− <42.5	4
42.5− <47.5	6
47.5− <52.5	1
52.5− <57.5	1

Example
Solution

Money	f_i	m_i	d_i	$f_i d_i$	$f_i d_i^2$
12.5− <17.5	2	15	-4	-8	32
17.5− <22.5	22	20	-3	-66	198
22.5− <27.5	19	25	-2	-38	76
27.5− <32.5	14	30	-1	-14	14
32.5− <37.5	13	35	0	0	0
37.5− <42.5	4	40	1	4	4
42.5− <47.5	6	45	2	12	24
47.5− <52.5	1	50	3	3	9
52.5− <57.5	1	55	4	4	16
	72			-103	373

Example

Solution⇒Cont...

$$
\begin{aligned}
S^2 &= \left[\frac{\sum_{i=1}^{k} f_i d_i^2 - \frac{\left(\sum_{i=1}^{k} f_i d_i\right)^2}{n}}{(n-1)}\right] w^2 \\
&= \left[\frac{373 - \frac{(-103)^2}{72}}{71}\right] \times 5^2 \\
&= \left[\frac{373 - \frac{10609}{72}}{71}\right] \times 25 \\
&= 79.455
\end{aligned}
$$

[4] Inter Quartile Range (IQR)

- The interquartile range (IQR) is a descriptive statistic used to summarize the extent of the spread of your data.
- The IQR is the distance between the 1^{st} quartile (25^{th} percentile) and 3^{rd} quartile (75^{th} percentile),

$$\mathrm{IQR} = Q_3 - Q_1.$$

- Fifty percent of the measurements are between the lower quartile and the upper quartile.

[4] Inter Quartile Range (IQR)

Advantage and disadvantage

- **Advantage**
  More stable estimator of spread since they use two values closer to middle of the distribution that vary less from sample to sample than more extreme values.
- **Disadvantage**
  These measures are totally dependent on just two values and ignore all other observations in a data set.

[5] Semi Inter Quartile Range

- The semi Inter quartile range is a slightly better measure of absolute dispersion than the range.
- But it ignores the observation on the tails.

$$\begin{aligned}\text{Semi Inter Quartile Range} &= \text{Quartile deviation}\\ &= \frac{Q_3 - Q_1}{2}.\end{aligned}$$

[6] Coefficient of Quartile Deviation

- A relative measure of dispersion based on the quartile deviation is called the coefficient of quartile deviation. It is defined as,

$$\text{Coefficient of Quartile Deviation} = \frac{\frac{Q_3 - Q_1}{2}}{\frac{Q_3 + Q_1}{2}} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

- It is a pure number that is free from any units of measurement. It can be used for comparing the dispersion in two or more than two sets of data.

Example

The wheat production (in Kg) of 20 acres is given as:

1120, 1240, 1320, 1040, 1080, 1200, 1440, 1360, 1680, 1730, 1785, 1342, 1960, 1880, 1755, 1720, 1600, 1470, 1750, 1885.

Find the IQR, quartile deviation and coefficient of quartile deviation.

Example
Solution

After arranging the observations in ascending order, we get

1040, 1080, 1120, 1200, 1240, 1320, 1342, 1360, 1440, 1470, 1600, 1680, 1720, 1730, 1750, 1755, 1785, 1880, 1885, 1960.

$$Q_1 = \text{Value of } \left[\frac{n+1}{4}\right]^{th} \text{ item}$$

$$Q_1 = \text{Value of } \left[\frac{20+1}{4}\right]^{th} \text{ item}$$

Example

Solution⇒Cont...

$$
\begin{aligned}
Q_1 &= \text{Value of } [5.25]^{th} \text{ item} \\
Q_1 &= 5^{th} \text{ item} + 0.25(6^{th} \text{ item} - 5^{th} \text{ item}) \\
&= 1240 + 0.25(1320 - 1240) \\
&= 1240 + 20 \\
&= 1260
\end{aligned}
$$

Example
Solution⇒Cont...

$$
\begin{aligned}
Q_3 &= \text{Value of } \ 3\left[\frac{n+1}{4}\right]^{th} \text{ item} \\
Q_3 &= \text{Value of } \ 3\left[\frac{20+1}{4}\right]^{th} \text{ item} \\
Q_3 &= \text{Value of } \ [15.75]^{th} \text{ item} \\
&= \ 15^{th} \text{ item} + 0.75(16^{th} \text{ item} - 15^{th} \text{ item}) \\
&= 1750 + 0.75(1755 - 1750) \\
&= 1753.75
\end{aligned}
$$

Example
Solution⇒Cont...

$$\begin{aligned}
\mathrm{IQR} &= Q_3 - Q_1 = 1753.75 - 1260 = 492.75 \\
\mathrm{QD} &= \frac{Q_3 - Q_1}{2} = \frac{492.75}{2} = 246.875 \\
\text{Coefficient of QD} &= \frac{Q_3 - Q_1}{Q_3 + Q_1} \\
&= \frac{1753.75 - 1260.00}{1753.75 + 1260.00} \\
&= 0.164
\end{aligned}$$

[7] Five number summary

The five number summary of a set of observations on a single variable consists of the following statistics:

1. Maximum (max)
2. Upper Quartile (Q3)
3. Median (M)
4. Lower Quartile (Q1)
5. Minimum (min)

Example

Compute the five number summary for the following observations:

19 11 7 24 13 15 10 3 10 20.

Example

Solution

- We order the observations 3 7 10 10 11 13 15 19 20 24.
- The minimum and maximum are 3 and 24, respectively.
- The median is $(11 + 13)/2 = 12$ because 11 and 13 are the two observations in the middle of the list.
- The lower quartile is 9.25.
- The upper quartile is 19.25.

Graphical representation of variation of data

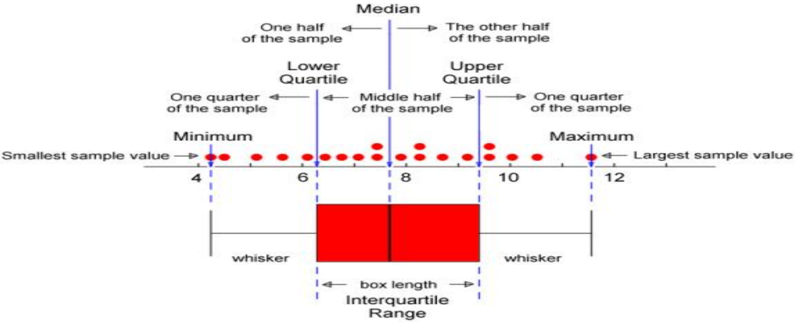
1. Box-and-whisker plot
2. Stem and leaf plot

[1] Box-and-whisker plot

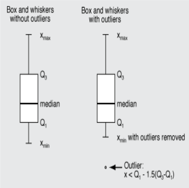
- Using box-and-whisker plot we can represent five-number summary visually.
- The length of the box is the interquartile range of the sample.
- A line is drawn across the box at the sample median.
- Whiskers sprout from the two ends of the box until they reach the sample maximum and minimum.

[1] Box-and-whisker plot

Cont...



- An outlier is any value that lies more than one and a half times the length of the box from either end of the box.
- That is, if a data point is below $Q_1 - 1.5\times$IQR or above $Q_3 + 1.5\times$ IQR, it is viewed as being too far from the central values to be reasonable.



Example

Find the outliers, if any, for the following data set:

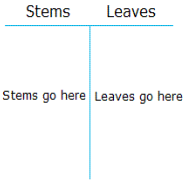
10.2, 14.1, 14.4. 14.4, 14.4, 14.5, 14.5, 14.6, 14.7, 14.7, 14.7, 14.9, 15.1, 15.9, 16.4.

Example
Solution

1. The median will be at position $(15 + 1) \div 2 = 8$. Then $Q_2 = 14.6$.
2. $Q_1 = 14.4$ and $Q_3 = 14.9$.
3. Then $\text{IQR} = 14.9 - 14.4 = 0.5$.
4. Outliers will be any points below Q_1 - $1.5\times\text{IQR} = 14.4$ - $0.75 = 13.65$ or above $Q_3 + 1.5\times\text{IQR} = 14.9 + 0.75 = 15.65$.
5. Then the outliers are at 10.2, 15.9, and 16.4.

[2] Stem and leaf plot

- A stem and leaf plot organizes data by showing the items in order using stems and leaves.
- The leaf is the last digit on the right or the ones digits. The stem is the remaining digit or digits.
- For 12, 2 is the leaf and 1 is the stem.
- For 45.7, 7 is the leaf and 45 is the stem.



Example 1

Draw the stem and leaf plot for following data.
24, 10, 13, 2, 28, 34, 65, 67, 55, 34, 25, 59, 6, 39, 61.

Solution

1. First, put this data in order 2, 6, 10, 13, 24, 25, 28, 34, 34, 39, 55, 59, 61, 65, 67.
2. We will use 0, 1, 2, 3, 4, 5, and 6 as stems.

Stems	Leaves
0	2 6
1	0 3
2	4 5 8
3	4 4 9
4	
5	5 9
6	1 5 7

Means 25

Means 55

Example 2

Draw the stem and leaf plot for following data.
104, 107, 112, 115, 115, 116, 123, 130, 134, 145, 147.

Solution

1. This time, the data is already in order.
2. We will use 10, 11, 12, 13, and 14 as stems.

Stems	Leaves			
10	4	7		
11	2	5	5	6
12	3			
13	0	4		
14	5	7		

Means 145

Example 3

Draw the stem and leaf plot for following two groups.

$$\text{Grade for class A} \quad : \quad 60, 68, 70, 75, 84, 86, 90, 91, 92, 94, 94, 96, 100, 100$$

$$\text{Grade for class B} \quad : \quad 60, 60, 70, 71, 73, 73, 75, 76, 77, 84, 85, 86, 91, 92$$

The plot is displayed as:

Class A Leaves	Stems	Class B Leaves
8 0	6	0 0
5 0	7	0 1 3 3 5 6 7
6 4	8	4 5 6
6 4 4 2 1 0	9	1 2
0 0	10	

Pass paper 2011

(i) Suppose the following pollution levels are observed in a river:
1.2, 1.4, 2.3, 2.5, 2.6, 3.4, 3.4, 3.8, 5.2, 5.6
Draw stem and leaf plot for above data.

(ii) Following data represent the amount of daily expenditure for a family in 17 different days.
100, 110, 200, 220, 220, 240, 250, 310, 340, 360, 400, 460, 470, 470, 470, 510, 510
Draw stem and leaf plot for above data.

(i) This data set has one decimal place and the stem-and-leaf plot does not show decimals.
To show decimal notation, we will state as much in the key.

stem	leaf
1	2 4
2	3 5 6
3	4 4 8
4	
5	2 6

key:"5|2" means "5.2"

(ii) Every number in our data set ends in zero.
In this case, let's make the tens digit our leaf and our hundreds digit our stem.
Notice our key tells others how the data should be interpreted.

stem	leaf
1	0 1
2	0 2 2 4 5
3	1 4 6
4	0 6 7 7 7
5	1 1

key:"5|1" means "510"

Thank You