Applied Statistics I

(IMT224β/AMT224β)

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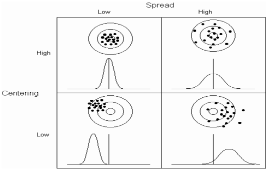
Measures of central tendency

Introduction

- We can characterize any set of data by measuring its central tendency, variation, and shape.
- A measure of central tendency is a single value that attempts to describe a set of data by identifying the central position within that set of data.



Popular measures of central tendency

Some popular measures of central tendency are:

1. Mean
2. Weighted mean
3. Median
4. Mode
5. Quartiles
6. Percentile

[1] Mean

- The **mean** is the most common measure of central tendency.
- The mean serves as a balance point in a set of data.
- You can calculate the mean by adding together all the values in a data set and then dividing that sum by the number of values in the data set.

[1] Mean

Population mean vs sample mean

- If the population consists of N observations $x_1, x_2, ..., x_N$ then **population mean** μ can be defined as:

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}.$$

- If the sample consists of n observations $x_1, x_2, ..., x_n$ then **sample mean** $\overline{x}$ can be defined as:

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

The sodium content of five different cans of soup are measured as 108.6, 104.2, 96.1, 99.6, and 102.2 mg. Determine the mean amount of sodium content.

$$
\begin{aligned}
\overline{x} &= \frac{\text{Sum of the values}}{\text{Number of values}} \\
\overline{x} &= \frac{\sum_{i=1}^{n} x_i}{n} \\
\overline{x} &= \frac{108.6 + 104.2 + 96.1 + 99.6 + 102.2}{5} \\
\overline{x} &= 102.1
\end{aligned}
$$

So the mean amount of sodium content is 102.1 mg.

Suppose you define the time to get ready as the time in minute from when you get out of the bed to when you leave your home. You collect the times shown below for 10 consecutive works days. Calculate the mean time.

Day	1	2	3	4	5	6	7	8	9	10
Time	39	29	43	52	39	44	40	31	44	35

$$\bar{x} = \frac{\text{Sum of the values}}{\text{Number of values}}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{x} = \frac{39 + 29 + 43 + 52 + 39 + 44 + 40 + 31 + 44 + 35}{10}$$

$$\bar{x} = \frac{396}{10} = 39.6$$

The mean time is 39.6 minutes.

(a) Mean of summarized data without class intervals

(b) Mean of summarized data with class intervals

(a) Mean of summarized data without class intervals

The mean when data are summarized with frequencies are given by

$$\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i}, \text{ k=no of values.}$$

(a) Mean of summarized data without class intervals

Example 1

Consider the frequency distribution shown in scores of 20 students in a science test. Find the mean marks of the students?

Marks(x)	Frequency(f)
40	1
50	2
60	4
70	3
80	5
90	2
100	3
Total	20

(a) Mean of summarized data without class intervals

Example 1⇒Solution

Marks(x)	Frequency(f)	fx
40	1	40
50	2	100
60	4	240
70	3	210
80	5	400
90	2	180
100	3	300
Total	20	1470

$$\text{Mean marks} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i} = \frac{1470}{20} = 73.5$$

(a) Mean of summarized data without class intervals

Example 2

The number of telephone calls received by a company switchboard over 5 minute intervals is given in the below table.

Number of Calls (x)	0	1	2	3	4	5
Frequency (f)	7	10	15	29	13	6

Find the mean number of telephone calls received by the company switchboard over 5 minute intervals.

(a) Mean of summarized data without class intervals

Example 2⇒Solution

Number of Calls (x)	Frequency (f)	fx
0	7	0
1	10	10
2	15	30
3	29	87
4	13	52
5	6	30
Total	80	209

$$\text{Mean number of calls} = \frac{\sum_{i=1}^{k} f_i x_i}{\sum_{i=1}^{k} f_i} = \frac{209}{80} = 2.6125$$

(b) Mean of summarized data with class intervals

If we have summarized data with classes, we can use two methods to find mean.

(i) Direct method.

(ii) Step deviation method.

(i) Direct method

$$\begin{aligned} \text{Mean} &= \frac{\sum_{i=1}^{k} f_i m_i}{\sum_{i=1}^{k} f_i}; \\ k &= \text{number of classes} \\ f_i &= \text{frequency of } i^{th} \text{ class} \\ m_i &= \text{mid value of } i^{th} \text{ class} \end{aligned}$$

(i) Direct method

Example

Compute the mean of the given data set.

Weight(Kg)	f_i
50.5− <53.5	1
53.5− <56.5	2
56.5− <59.5	6
59.5− <62.5	11
62.5− <65.5	16
65.5− <68.5	9
68.5− <71.5	4
71.5− <74.5	1

(i) Direct method

Example⇒Solution

Weight(Kg)	f_i	m_i	$f_i m_i$
50.5− <53.5	1	52	52
53.5− <56.5	2	55	110
56.5− <59.5	6	58	348
59.5− <62.5	11	61	671
62.5− <65.5	16	64	1024
65.5− <68.5	9	67	603
68.5− <71.5	4	70	280
71.5− <74.5	1	73	73
	50		3161

$$\bar{x} = \frac{\sum_{i=1}^{8} f_i m_i}{\sum_{i=1}^{8} f_i} = \frac{3163}{50} = 63.22$$

(ii) Step deviation method

$$\bar{x} = A + \left(\frac{\sum_{i=1}^{k} f_i d_i}{\sum_{i=1}^{k} f_i} \right) w;$$

k = number of classes

A = assumed mean

w = width of the class intervals that it lies

d_i = deviation of the i^{th} class from the class that it lies.

(ii) Step deviation method

Example

The emission of sulfur oxide of an industrial plant at 80 determinations are recorded and summarized by the following table.

Amount	f_i
$5- <9$	3
$9- <13$	10
$13- <17$	14
$17- <21$	25
$21- <25$	17
$25- <29$	9
$29- <33$	2

Find the mean.

(ii) Step deviation method

Example⇒Solution

Amount	f_i	m_i	d_i	$f_i d_i$
5− <9	3	7	-3	-9
9− <13	10	11	-2	-20
13− <17	14	15	-1	-14
17− <21	25	**19**	0	0
21− <25	17	23	1	17
25− <29	9	27	2	18
29− <33	2	31	3	6
	80			-2

Let $A = 19$

Let $17- < 21$ be the class interval that A lies

$$\begin{aligned} \overline{x} &= 19 + \left[\frac{(-2).4}{80}\right] \\ &= 18.9 \end{aligned}$$

- Mean always exists and it is unique.
- Mean depends on extreme values.
- It takes into account every item of data.

[2] Weighted mean

- Arithmetic mean computed by considering relative importance of each item is called as weighted mean.
- Instead of each of the data points contributing equally to the final average, some data points contribute more than others.

[2] Weighted mean

Cont...

If $x_1, x_2, ..., x_n$ are values, whose relative importance is expressed numerically by a corresponding set of numbers $w_1, w_2, ..., w_n$, then weighted mean $\overline{x}_w$, is given by,

$$\overline{x}_w = \frac{x_1 w_1 + x_2 w_2 + ... + x_n w_n}{w_1 + w_2 + ... + w_n}.$$

[2] Weighted mean
Cont...

If all the weights are equal, then the weighted mean is the same as the arithmetic mean.

$$\bar{x}_w = \frac{x_1 w_1 + x_2 w_2 + \ldots + x_n w_n}{w_1 + w_2 + \ldots + w_n}$$

$$\bar{x}_w = \frac{x_1 w + x_2 w + \ldots + x_n w}{w + w + \ldots + w}, \quad (\text{since } w_i = w)$$

$$\bar{x}_w = \frac{w(x_1 + x_2 + \ldots + x_n)}{nw}$$

$$\bar{x}_w = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

$$\bar{x}_w = \bar{x}$$

[2] Weighted mean
Example

A student obtained $40, 50, 60, 80$, and 45 marks in the subjects of Mathematics, Statistics, Physics, Chemistry and Biology respectively. Assuming weights $5, 2, 4, 3$, and 1 respectively for the above mentioned subjects. Find weighted mean.

$$
\begin{aligned}
\overline{x}_w &= \frac{x_1 w_1 + x_2 w_2 + \ldots + x_n w_n}{w_1 + w_2 + \ldots + w_n} \\
&= \frac{40 \times 5 + 50 \times 2 + 60 \times 4 + 80 \times 3 + 45 \times 1}{5 + 2 + 4 + 3 + 1} \\
&= \frac{825}{15} \\
&= 55 \text{ marks/subject.}
\end{aligned}
$$

[3] Median

- The median of a finite list of numbers can be found by arranging all the observations from lowest value to highest value and picking the middle one.
- If the total number of values in the sample (or population) is even, then the median is the mean of the two middle numbers.

[3] Median
Example 1

Find the median of the values $4, 1, 8, 13, 11$

Solution

Arrange data $\Rightarrow 1, 4, 8, 11, 13$

$$
\begin{aligned}
\text{Median} &= \text{Value of } \left[\frac{n+1}{2}\right]^{th} \text{ item} \\
\text{Median} &= \text{Value of } \frac{6}{2} \text{ item} = 3^{rd} \text{ item} \\
\text{Median} &= 8
\end{aligned}
$$

Find the median of the values $5, 7, 10, 20, 16, 12$.

Solution

Arrange data $\Rightarrow 5, 7, 10, 12, 16, 20$

$$\begin{aligned}
\text{Median} &= \text{Value of } \left[\frac{n+1}{2}\right]^{th} \text{ item} \\
\text{Median} &= \frac{7}{2}^{\text{th}} \text{ item} \\
&= 3.5^{\text{th}} \text{ item} \\
&= \frac{10+12}{2} = 11
\end{aligned}$$

Find the median of the following data set.

Weight (Kg)	No of students
20	5
22	7
23	4
24	1
27	3
28	4
30	1

[3] Median

Median of summarized data without class intervals⇒Cont...

Weight (Kg)	No of students	Cumulative frequency
20	5	5
22	7	12
23	4	16
24	1	17
27	3	20
28	4	24
30	1	25

$$\begin{aligned} n &= 25 \\ \left(\frac{n+1}{2}\right) &= \left(\frac{26}{2}\right) = 13 \\ \text{Median} &= 23 \text{ (value of } 13^{th} \text{ item)} \end{aligned}$$

- We use following formula to find the median,

$$\begin{aligned}
\text{Median} &= L_i + \left(\frac{n}{2} - c_{i-1}\right)\frac{w}{f_i}, \\
i &= \text{median class} \\
L_i &= \text{lower boundary of the median class} \\
w &= \text{class width of median class} \\
f_i &= \text{frequency of median class} \\
c_{i-1} &= \text{cumulative frequency of } (i-1)^{\text{th}} \text{ class.}
\end{aligned}$$

Median of summarized data with class intervals
Example 1

Calculate median from the following data.

Age	Frequency
20-25	2
25-30	14
30-35	29
35-40	43
40-45	33
Over 45	9

Median of summarized data with class intervals

Example 1⇒Solution

Age	Frequency	Cumulative frequency
20-25	2	2
25-30	14	16
30-35	29	45
35-40	43	88
40-45	33	121
Over 45	9	130

Median of summarized data with class intervals

Example 1⇒Solution⇒Cont...

$$\begin{aligned}
\left(\frac{n+1}{2}\right)^{th} \text{ item} &= \frac{130+1}{2} = 65.5^{th} \text{ item} \\
\text{Median class interval} &\Rightarrow 35-40 \\
\text{Median} &= L_i + \left(\frac{n}{2} - c_{i-1}\right)\frac{w}{f_i}. \\
&= 35 + \frac{5}{43}(65-45) \\
&= 35 + \frac{5}{43}(20) \\
&= 37.33
\end{aligned}$$

Median of summarized data with class intervals

Example 2

Calculate median from the following data.

Group	Frequency
60-64	1
65-69	5
70-74	9
75-79	12
80-84	7
85-89	2

Median of summarized data with class intervals

Example 2⇒Solution

Group	Frequency	Class boundary	Cumulative frequency
60-64	1	59.5 - 64.5	1
65-69	5	64.5 - 69.5	6
70-74	9	69.5 - 74.5	15
75-79	12	74.5 - 79.5	27
80-84	7	79.5 - 84.5	34
85-89	2	84.5 - 89.5	36

Median of summarized data with class intervals

Example 2⇒Solution⇒Cont...

$$
\begin{aligned}
\left(\frac{n+1}{2}\right)^{th} \text{ item} &= \frac{36+1}{2} = 18.5^{th} \text{ item} \\
\text{Median class interval} &\Rightarrow 74.5 - 79.5 \\
\text{Median} &= L_i + \left(\frac{n}{2} - c_{i-1}\right)\frac{w}{f_i}. \\
&= 74.5 + \frac{5}{12}(18 - 15) \\
&= 74.5 + \frac{5}{12}(3) \\
&= 74.5 + 1.25 \\
&= 75.75
\end{aligned}
$$

- It is unique and exists always.
- Has no effect from extreme values of the data set. Therefore, median is a useful number in cases where the distribution has very large extreme values which would otherwise skew the data.
- It is not necessarily a particular observation of the data set.

[4] Mode

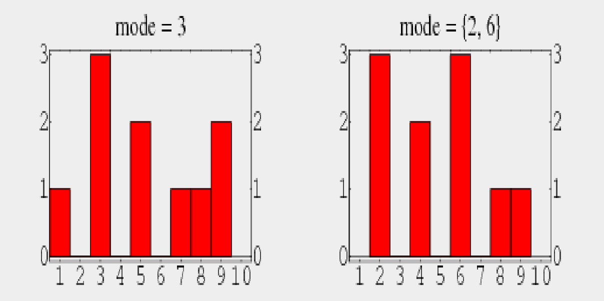
- The mode is the value that occurs most frequently in a data set.
- There may be more than one mode when two or more numbers have an equal number of instances and this is also the maximum instances.
- A mode does not exist if no number has more than one instance.

[4] Mode
Cont...

- A distribution with a single mode is said to be unimodal.
- A distribution with two modes is said to be bimodal.
- A distribution with more than two modes is said to be multimodal.

[4] Mode
Example 1

- For a data set, $3, 7, 3, 9, 9, 3, 5, 1, 8, 5$ the unique mode is 3 (left histogram).
- For a data set, $2, 4, 9, 6, 4, 6, 6, 2, 8, 2$ there are two modes: 2 and 6 (right histogram).



Find the mode of the following data set.

Values	f_i
3	5
5	2
6	8
8	4
11	3

Solution

For summarized data set with values, the mode is the most frequently occurring value.

Therefore mode is 6.

When the data are summarized with class intervals, the mode is given by

$$\begin{aligned}
\text{Mode} &= L_i + \left[\frac{(f_i - f_{i-1})}{(f_i - f_{i-1}) + (f_i - f_{i+1})}\right] w \\
i &= \text{modal class} \\
L_i &= \text{lower boundary of modal class} \\
f_i &= \text{frequency of modal class} \\
f_{i-1} &= \text{frequency of } (i-1)^{th} \text{ class} \\
f_{i+1} &= \text{frequency of } (i+1)^{th} \text{ class} \\
w &= \text{class width of modal class.}
\end{aligned}$$

Calculate the mode of the given summarized data set.

Age	No of people
$20- < 25$	60
$25- < 30$	80
$30- < 35$	100
$35- < 40$	180
$40- < 45$	150
$45- < 50$	80
$50- < 55$	120
$55- < 60$	90

Mode of summarized data with class intervals

Example⇒Solution

Mode class is $35- < 40$.

$$
\begin{aligned}
\text{Mode} &= L_i + \left[\frac{(f_i - f_{i-1})}{(f_i - f_{i-1}) + (f_i - f_{i+1})}\right] w \\
L_i &= 35 \\
f_i &= 180 \\
f_{i-1} &= 100 \\
f_{i+1} &= 150 \\
w &= 5. \\
\text{Mode} &= 35 + \left[\frac{(180 - 100)}{(180 - 100) + (180 - 150)}\right] 5 \\
&= 35 + \frac{80}{110} \times 5 \\
&= 38.635
\end{aligned}
$$

Relationship between mean, median and mode

An interesting empirical relationship between the sample mean, statistical median, and mode which appears to hold for unimodal curves of moderate asymmetry is given by

$$\text{mode} \simeq \text{mean-3(mean-median)}.$$

(a) For moderately skewed distribution mode=50.04, mean=45. Find median.

(b) If medain=20, and mean=22.5 in moderately skewed distribution then compute approximate value mode.

Relationship between mean, median and mode

Example⇒Solution

(a)

$$\begin{aligned} \text{mode} &\simeq \text{mean-3(mean-median)} \\ 50.04 &\simeq 45 - 3(45 - \text{median}) \\ \text{median} &\simeq 46.68 \end{aligned}$$

(b)

$$\begin{aligned} \text{mode} &\simeq \text{mean-3(mean-median)} \\ \text{mode} &\simeq 22.5 - 3(22.5 - 20) \\ \text{mode} &\simeq 15 \end{aligned}$$

[5] Quartiles

- There are three quartiles called, first quartile, second quartile and third quartile.
- There quartiles divides the set of observations into four equal parts.
- The second quartile is equal to the median.
- The first quartile is also called lower quartile and is denoted by Q_1.

[5] Quartiles
Cont...

- The third quartile is also called upper quartile and is denoted by Q_3.
- The lower quartile Q_1 is a point which has 25% observations less than it and 75% observations are above it.
- The upper quartile Q_3 is a point with 75% observations below it and 25% observations above it.

[5] Quartiles
Cont...

$$
\begin{aligned}
Q_1 &= \text{Value of } \left[\frac{n+1}{4}\right]^{th} \text{ item} \\
Q_2 &= \text{Value of } 2\left[\frac{n+1}{4}\right]^{th} \text{ item} = \text{Median} \\
Q_3 &= \text{Value of } 3\left[\frac{n+1}{4}\right]^{th} \text{ item}
\end{aligned}
$$

The wheat production (in Kg) of 20 acres is given as:

1120, 1240, 1320, 1040, 1080, 1200, 1440, 1360, 1680, 1730, 1785, 1342, 1960, 1880, 1755, 1720, 1600, 1470, 1750, 1885.

Find Q_1 and Q_3.

After arranging the observations in ascending order, we get

1040, 1080, 1120, 1200, 1240, 1320, 1342, 1360, 1440, 1470,
1600, 1680, 1720, 1730, 1750, 1755, 1785, 1880, 1885, 1960.

$$Q_1 = \text{Value of } \left[\frac{n+1}{4}\right]^{th} \text{ item}$$

$$Q_1 = \text{Value of } \left[\frac{20+1}{4}\right]^{th} \text{ item}$$

[5] Quartiles

Example⇒Solution⇒Cont...

$$
\begin{aligned}
Q_1 &= \text{Value of } [5.25]^{th} \text{ item} \\
Q_1 &= 5^{th} \text{ item} + 0.25(6^{th} \text{ item} - 5^{th} \text{ item}) \\
&= 1240 + 0.25(1320 - 1240) \\
&= 1240 + 20 \\
&= 1260
\end{aligned}
$$

$$
\begin{aligned}
Q_3 &= \text{Value of } \; 3\left[\frac{n+1}{4}\right]^{th} \text{ item} \\
Q_3 &= \text{Value of } \; 3\left[\frac{20+1}{4}\right]^{th} \text{ item} \\
Q_3 &= \text{Value of } \; [15.75]^{th} \text{ item} \\
&= 15^{th} \text{ item} + 0.75(16^{th} \text{ item} - 15^{th} \text{ item}) \\
&= 1750 + 0.75(1755 - 1750) \\
&= 1753.75
\end{aligned}
$$

The following table shows the distribution of 128 families according to the number of children.

No of children	No of families
0	20
1	15
2	25
3	30
4	18
5	10
6	6
7	3
8 or more	1

Find the quantile.

Quartiles of summarized data without class intervals

Example⇒Solution

No of children	No of families	Cumulative frequency
0	20	20
1	15	35
2	25	60
3	30	90
4	18	108
5	10	118
6	6	124
7	3	127
8 or more	1	128

Quartiles of summarized data without class intervals

Example⇒Solution⇒Cont...

$$
\begin{aligned}
Q_1 &= \left(\frac{128+1}{4}\right)^{th} \text{ observation} \\
&= (32.25)^{th} \text{ observation} \\
&= 1 \\
Q_2 &= \left(\frac{128+1}{2}\right)^{th} \text{ observation} \\
&= (64.5)^{th} \text{ observation} \\
&= 3 \\
Q_3 &= 3\left(\frac{128+1}{4}\right)^{th} \text{ observation} \\
&= (96.75)^{th} \text{ observation} \\
&= 4
\end{aligned}
$$

Quartiles of summarized data with class intervals

First quartile

$$
\begin{aligned}
Q_1 &= L_i + \left(\frac{n}{4} - c_{i-1}\right)\frac{w}{f_i}; \\
L_i &= \text{lower boundary of the class in which } Q_1 \text{ lies} \\
f_i &= \text{frequency of that class} \\
w &= \text{width of that class} \\
c_{i-1} &= \text{cumulative frequency of proceeding class.}
\end{aligned}
$$

Quartiles of summarized data with class intervals

Second quartile

$$
\begin{aligned}
Q_2 &= L_i + \left(\frac{n}{2} - c_{i-1}\right)\frac{w}{f_i}; \\
L_i &= \text{lower boundary of the class in which } Q_2 \text{ lies} \\
f_i &= \text{frequency of that class} \\
w &= \text{width of that class} \\
c_{i-1} &= \text{cumulative frequency of proceeding class.}
\end{aligned}
$$

Quartiles of summarized data with class intervals
Third quartile

$$
\begin{aligned}
Q_3 &= L_i + \left(\frac{3n}{4} - c_{i-1}\right)\frac{w}{f_i}; \\
L_i &= \text{lower boundary of the class in which } Q_3 \text{ lies} \\
f_i &= \text{frequency of that class} \\
w &= \text{width of that class} \\
c_{i-1} &= \text{cumulative frequency of proceeding class.}
\end{aligned}
$$

Quartiles of summarized data with class intervals

Example

Calculate the quartile from the data given below:

Maximum Load	Number of Cables
9.3-9.7	2
9.8-10.2	5
10.3-10.7	12
10.8-11.2	17
11.3-11.7	14
11.8-12.2	6
12.3-12.7	3
12.8-13.2	1

Quartiles of summarized data with class intervals

Example⇒Solution

Maximum Load	No of Cables	Class boundary	C.F
9.3-9.7	2	9.25-9.75	2
9.8-10.2	5	9.75-10.25	7
10.3-10.7	12	10.25-10.75	19
10.8-11.2	17	10.75-11.25	36
11.3-11.7	14	11.25-11.75	50
11.8-12.2	6	11.75-12.25	56
12.3-12.7	3	12.25-12.75	59
12.8-13.2	1	12.75-13.25	60

$$
\begin{aligned}
Q_1 &= \text{value of } \left(\frac{n+1}{4}\right)^{th} \text{ item} \\
Q_1 &= \text{value of } \left(\frac{60+1}{4}\right)^{th} \text{ item} \\
&= 15.25^{th} \text{ item} \\
Q_1 &\Rightarrow \text{ lies in the class } 10.25 - 10.75 \\
Q_1 &= 10.25 + (15-7)\frac{0.5}{12} \\
&= 10.25 + 0.33 = 10.58
\end{aligned}
$$

Quartiles of summarized data with class intervals

Example⇒Solution⇒Cont...

$$
\begin{aligned}
Q_3 &= \text{value of } \left(\frac{3(n+1)}{4}\right)^{th} \text{ item} \\
Q_3 &= \text{value of } \left(\frac{3 \times 61}{4}\right)^{th} \text{ item} \\
&= 45.75^{th} \text{ item} \\
Q_3 &\Rightarrow \text{ lies in the class } 11.25 - 11.75 \\
Q_3 &= 11.25 + (45 - 36)\,\frac{0.5}{14} \\
&= 11.25 + 0.32 = 11.57
\end{aligned}
$$

[6] Percentile

- A percentile is the value of a variable below which a certain percent of observations fall.
- For example, the 20^{th} percentile is the value (or score) below which 20 percent of the observations may be found.

Consider the marks of students for MCQ paper of 40 questions. The marks are recorded as followings.

Marks	Number of Students
0-5	3
6-10	10
11-15	14
16-20	20
21-25	13
26-30	9
31-35	1

Find quartiles and 45^{th} percentile.

In finding percentiles we use the graph of percentage cumulative frequency polygon.

Marks	Number of Students	CF
0-5	3	3
6-10	10	13
11-15	14	27
16-20	20	47
21-25	13	60
26-30	9	69
31-35	1	70

Left class boundaries	CF	CF%
0	0	0.00
5.5	3	4.28
10.5	13	18.57
15.5	27	38.57
20.5	47	67.14
25.5	60	85.71
30.5	69	98.57
35.5	70	100.00

Thank You