Mathematics for Biology

MAT1142

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Real World Application of Mathematics

- A quantity is said to be subject to **exponential growth** if it **increases** at a rate proportional to its value.
- A quantity is said to be subject to **exponential decay** if it **decreases** at a rate proportional to its value.
- Symbolically, those processes can be modeled by the following **differential equation**, where P is the quantity and r is a constant.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP.$$

The solution to above equation is,

$$P(t) = P_0 e^{rt}$$

- P_0 is the original amount.
- t is the amount of time.
- $P(t)$ is the amount at time t.
- r is a constant representing the **growth rate** or **decreasing rate**.
- If $r > 0$, the function models a growing situation.
- If $r < 0$, the function models a decreasing situation.

Exponential Growth and Decay Model

Graph of Exponential Growth and Decay Model

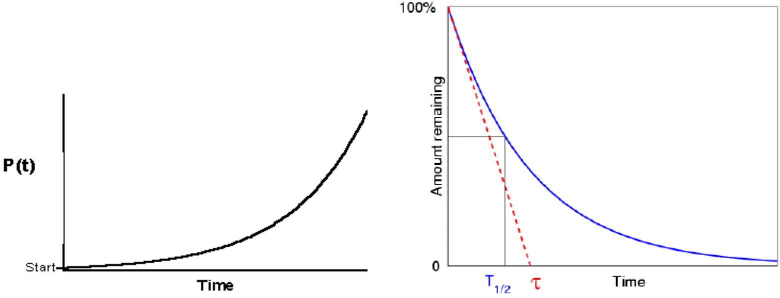

Figure: Exponential growth and decay.

Suppose that the population of a certain country grows at an annual rate of 2%. If the current population is 3 million, what will the population be in 10 years?

Solution

If we measure population in millions and time in years, then $P(t) = P_0 e^{rt}$ with $P_0 = 3$ and $r = 0.02$. Inserting these particular values into formula

$$P(t) = 3e^{0.02t}$$

The population in 10 years is $P(10) = P(t) = 3e^{0.02*10} \simeq 3$. 664208 million.

In the same country as in Example 1, how long will it take the population to reach 5 million?

Solution

$$P(t) = 3e^{0.02t}$$

Now we want to know when the future value $P(t)$ of the population at some time t will equal 5 million. Therefore, we need to solve the equation $P(t) = 5$ for time t.

Now we have the exponential equation $5 = 3e^{0.02t}$,

$$\begin{aligned}
\frac{5}{3} &= e^{0.02t} \\
\ln\frac{5}{3} &= \ln e^{0.02t} \\
\ln\frac{5}{3} &= 0.02t \\
t &= \frac{\ln\frac{5}{3}}{0.02} \\
t &\simeq 25.54128.
\end{aligned}$$

Thus, it would take about 25.54 years for the population to reach 5 million.

Suppose that a size of a bacterial culture is given by the function

$$P(t) = 100e^{0.15t}$$

where the size $P(t)$ is measured in grams and time t is measured in hours. How long will it take for the culture to double in size?

Solution

The initial size is $P_0 = 100$ grams, so we want to know when the future value $P(t)$ at some time t will equal 200. Therefore, we need to solve the equation $P(t) = 200$ for time t.

Now we have the exponential equation $200 = 100e^{0.15t}$. Using the same procedure as in the last example,

$$\begin{aligned}
\frac{200}{100} &= e^{0.15t} \\
2 &= e^{0.15t} \\
\ln(2) &= \ln e^{0.15t} \\
\ln(2) &= 0.15t \\
t &= \frac{\ln 2}{0.15} \\
t &\simeq 4.620981.
\end{aligned}$$

Thus, it would take about 4.62 hours for the size to double.

Suppose that a certain radioactive element has an annual **decay rate** of 10%. Starting with a 200 gram sample of the element, how many grams will be left in 3 years?

Solution

If we measuring size in grams and time in years, then $P(t) = P_0 e^{rt}$ with $P_0 = 200$ and $r = -0.10$. Inserting these particular values into formula

$$P(t) = 200e^{-0.10t}$$

The amount in 3 years is $P(3) = 200e^{-0.10*3} \simeq 148.1636$ grams.

Using the same element as in Example 4, if a particular sample of the element decays to 50 grams after 5 years, how big was the original sample?

Solution

This is a present value problem, where the unknown is the initial amount P_0. As before, $r = -0.10$, so

$$P(t) = P_0 e^{-0.10t}$$

Since $P(5) = 50$, we have the equation,

$$\begin{aligned} 50 = P(5) &= P_0 e^{-0.10*5} \\ P_0 &= \frac{50}{e^{-0.10*5}} \end{aligned}$$

So the original sample size $P_0 \simeq 82.43606$ grams.

Suppose that a certain radioactive isotope has an annual decay rate of 5%. How many years will it take for a 100 gram sample to decay to 40 grams?

Solution

Use $P(t) = P_0 e^{rt}$ with $P_0 = 100$ and $r = -0.05$, so

$$P(t) = 100e^{-0.05t}$$

Now we want to know when the future value $P(t)$ of the size of the sample at some time t will equal 40. Therefore, we need to solve the equation $P(t) = 40$ for time t, which leads to the exponential equation,

$$40 = 100e^{-0.05t}.$$

Using the procedure for solving exponential equations that was presented above,

$$\begin{aligned}
40 &= 100e^{-0.05t}\\
\frac{40}{100} &= e^{-0.05t}\\
0.4 &= e^{-0.05t}\\
\ln(0.4) &= \ln e^{-0.05t}\\
\ln(0.4) &= -0.05t\\
t &= \frac{\ln(0.4)}{-0.05}\\
t &\simeq 18.32581.
\end{aligned}$$

Thus, it would take approximately 18.33 years for the sample to decay to 40 grams.

Example 7

The time in which one-half the original amount decays is called as **half life**. Using the same element as in Example 6, what is the half life of the element?

Solution

As before, $r = -0.05$, so

$$P(t) = P_0 e^{-0.05t}.$$

The initial size is P_0 grams, so we want to know when the future value $P(t)$ at some time t will equal one-half the initial amount, $P_0/2$. Therefore, we need to solve the equation $P(t) = P_0/2$ for time t, which leads to the exponential equation,

$$\frac{P_0}{2} = P_0 e^{-0.05t}$$

Cont...

Using the same procedure as in the last example,

$$
\begin{aligned}
\frac{P_0}{2} &= P_0 e^{-0.05t} \\
\frac{1}{2} &= e^{-0.05t} \\
\ln\left(\frac{1}{2}\right) &= \ln e^{-0.05t} \\
\ln\left(\frac{1}{2}\right) &= -0.05t \\
t &= \frac{\ln\left(\frac{1}{2}\right)}{-0.05} \\
t &\simeq 13.86294.
\end{aligned}
$$

Thus, the half-life is approximately 13.86 years.

Newtons Law of Cooling

When a hot object is introduced into cool surroundings, the rate at which the object cools is not proportional to its temperature, but rather, is proportional to the difference in temperature between the object and the surroundings. Let $T(t)$ be the temperature of the object at the time t, T_0 be the initial temperature of the object, and T_s be the temperature of the surroundings. Then:

$$T'(t) = k(T(t) - T_s), \ T(0) = T_0.$$

Example 1

At 9:30am, a secret agent was found murdered in a 75°F room. The body temperature was 90°F and was 85°F at 10:00am. When was the time of death?

You have to assume that the normal body temperature of a adult human being as 98.6°F.

[You may assume that $\ln 1.573 = 0.453$ and $\ln(2/3) = -0.405$.]

Solution

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= k(T - T_s) \\
\frac{1}{T - T_s}\mathrm{d}T &= k\mathrm{d}t \\
\int \frac{1}{T - T_s}\mathrm{d}T &= \int k\mathrm{d}t \\
\ln|T - T_s| &= kt + c \\
T - T_s &= e^{kt+c} \\
T - T_s &= ce^{kt} \\
T(t) &= T_s + ce^{kt} \\
t = 0 \Rightarrow T(0) &= T_s + ce^{k0} \\
T_0 &= T_s + ce^{0} \\
T_0 &= T_s + c1
\end{aligned}
$$

Solution

$$\begin{aligned} c &= T_0 - T_s \\ T(t) &= T_s + (T_0 - T_s)e^{kt} \end{aligned}$$

We know that T_0=90°F, T_s=75°F, T(0.5)=85°F,
We need to find the time when T=98.6°F

$$\begin{aligned} T(t) &= 75 + (90 - 75)e^{kt} \\ T(t) &= 75 + 15e^{kt} \\ t = 0.5 \Rightarrow T(0.5) &= 75 + 15e^{k0.5} \\ 85 &= 75 + 15e^{k0.5} \\ 10 &= 15e^{0.5k} \end{aligned}$$

Solution

$$
\begin{aligned}
e^{0.5k} &= \frac{10}{15} \\
e^{0.5k} &= \frac{2}{3} \\
\ln e^{0.5k} &= \ln\left(\frac{2}{3}\right) \\
0.5k &= \ln\left(\frac{2}{3}\right) \\
k &= \frac{1}{0.5}\ln\left(\frac{2}{3}\right) \\
k &= 2\ln\left(\frac{2}{3}\right) \\
T(t) &= 75 + 15e^{2\ln(2/3)t} \\
T(t) &= 98.6
\end{aligned}
$$

Solution

$$
\begin{aligned}
75 + 15e^{2\ln(2/3)t} &= 98.6 \\
15e^{2\ln(2/3)t} &= 98.6 - 75 \\
15e^{2\ln(2/3)t} &= 23.6 \\
e^{2\ln(2/3)t} &= \frac{23.6}{15} \\
e^{2\ln(2/3)t} &= 1.573 \\
\ln e^{2\ln(2/3)t} &= \ln 1.573 \\
2\ln(2/3)t &= \ln 1.573
\end{aligned}
$$

Solution

$$
\begin{aligned}
t &= \frac{\ln 1.573}{2\ln(2/3)} \\
t &= \frac{0.453}{2 \times (-0.405)} \\
t &= -\frac{0.453}{0.810} \\
t &= -0.5592 \text{ hour} \\
t &= -0.5592(60) \\
t &= -33.55 \text{ minutes}
\end{aligned}
$$

The time of death is about 8.57 am.

Thank You