Real Analysis III

(MAT312β)

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The Gradient of a Scalar Field

What is gradient of a scalar field?

- Assume that there is a heat source in a room and the temperature does not change over time.
- Suppose the temperature in that room is given by a scalar field, f, so at each point (x, y, z) the temperature is $f(x, y, z)$.
- At each point in the room, the gradient of f at that point will show the direction the temperature rises most quickly.
- The magnitude of the gradient will determine how fast the temperature rises in that direction.

What is gradient of a scalar field?

The gradient of the function $f(x, y) = -(\cos^2 x + \cos^2 y)^2$

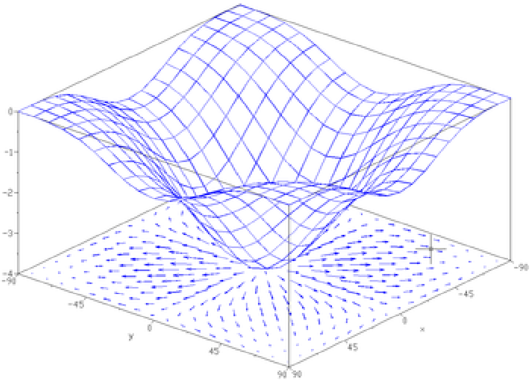

Figure: The gradient of the function $f(x, y) = -(\cos^2 x + \cos^2 y)^2$ depicted as a projected vector field on the bottom plane

Mathematical aspect of the gradient of a scalar field

- The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field.
- The magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field.
- If the vector is resolved, its components represent the rate of change of the scalar field with respect to each directional component.

Mathematical aspect of the gradient of a scalar field
Notations

- The gradient of a scalar field f is denoted ∇f.
- Where ∇ denotes the vector differential operator, del.
- The notation "grad(f)" is also commonly used for the gradient.
- Hence for a two-dimensional scalar field $f(x, y)$,

$$\operatorname{grad} f(x, y) = \nabla f(x, y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

- And for a three-dimensional scalar field $f(x, y, z)$,

$$\begin{aligned}\operatorname{grad} f(x,y,z) = \nabla f(x,y,z) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).\end{aligned}$$

- For a n-dimensional scalar field $f(x_1, x_2, ..., x_n)$,

$$\begin{aligned}\operatorname{grad} f(x_1, x_2, ..., x_n) = \nabla f &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right) f \\ &= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\right).\end{aligned}$$

In 2-space the gradient vector is often written as

$$\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x}\mathbf{i} + \frac{\partial f(x,y)}{\partial y}\mathbf{j}.$$

In 3-space the corresponding formula is

$$\nabla f(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\mathbf{i} + \frac{\partial f(x,y,z)}{\partial y}\mathbf{j} + \frac{\partial f(x,y,z)}{\partial z}\mathbf{k}.$$

In n-space the corresponding formula is

$$\begin{aligned}\nabla f(x_1,x_2,\ldots,x_n) &= \frac{\partial f(x_1,x_2,\ldots,x_n)}{\partial x_1}\mathbf{e}_1 + \frac{\partial f(x_1,x_2,\ldots,x_n)}{\partial x_2}\mathbf{e}_2 \\ &\quad + \ldots + \frac{\partial f(x_1,x_2,\ldots,x_n)}{\partial x_n}\mathbf{e}_n.\end{aligned}$$

Examples

For following scalar fields, calculate ∇f:

1. $f(x, y) = 8x + 5y$.
2. $f(x, y, z) = x^4 yz$.
3. $f(x, y) = x^2 \sin 5y$.

Given scalar field $f(x, y) = 8x + 5y$:

$$\begin{aligned} \nabla f(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= (8, 5). \end{aligned}$$

Given scalar field $f(x, y) = x^4yz$:

$$\begin{aligned} \nabla f(x,y,z) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(4x^3yz, x^4z, x^4y\right). \end{aligned}$$

Given scalar field $f(x,y) = x^2 \sin 5y$:

$$\begin{aligned}\nabla f(x,y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\\ &= (2x\sin(5y), 5x^2\cos(5y)).\end{aligned}$$

The first order Taylor formula using gradient

The formula in Theorem (6.1), which expresses $f'(\mathbf{a};\mathbf{y})$ as a linear combination of the components of $\mathbf{y}$, can now be written as a dot product,

$$f'(\mathbf{a};\mathbf{y}) = \sum_{k=1}^{n} D_k f(\mathbf{a}) y_k = \nabla f(\mathbf{a}).\mathbf{y}, \tag{1}$$

where $\nabla f(\mathbf{a})$ is the gradient of the scalar field f.

The first order Taylor formula using gradient
Cont...

- If f is a differentiable function at point $\mathbf{a}$ we have a Taylor formula,

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| E(\mathbf{a}, \mathbf{v}).$$

- From Theorem (6.1) we have

$$\begin{aligned} T_{\mathbf{a}}(\mathbf{y}) &= f'(\mathbf{a}; \mathbf{y}) \\ T_{\mathbf{a}}(\mathbf{v}) &= f'(\mathbf{a}; \mathbf{v}) = \nabla f(\mathbf{a}).\mathbf{v} \text{ (From (1))}. \end{aligned}$$

- The first order Taylor formula can now be written in the form

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + \nabla f(\mathbf{a}).\mathbf{v} + \|\mathbf{v}\| E(\mathbf{a}, \mathbf{v}), \tag{2}$$

where $E(\mathbf{a}, \mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$.

The first order Taylor formula using gradient
Cont...

- The above form of Taylor formula resembles the one-dimensional Taylor formula, with the gradient vector $\nabla f(\mathbf{a})$ playing the role of the derivative $f'(\mathbf{a})$.
- From the Taylor formula we can easily prove that differentiability implies continuity.

Theorem (7.1)

If a scalar field f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

From equation (2) we have

$$f(\mathbf{a}+\mathbf{v})-f(\mathbf{a}) \quad = \quad \nabla f(\mathbf{a}).\mathbf{v}+\|\mathbf{v}\|E(\mathbf{a},\mathbf{v})$$

By taking modulus from both side we have

$$|f(\mathbf{a}+\mathbf{v})-f(\mathbf{a})| \quad = \quad |\nabla f(\mathbf{a}).\mathbf{v}+\|\mathbf{v}\|E(\mathbf{a},\mathbf{v})|.$$

Applying the triangle inequality we find

$$0 \leq |f(\mathbf{a}+\mathbf{v})-f(\mathbf{a})| \quad \leq \quad |\nabla f(\mathbf{a}).\mathbf{v}|+|\|\mathbf{v}\|E(\mathbf{a},\mathbf{v})|.$$

Applying the Cauchy-Schwarz inequality we find

$$0 \leq |f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| \leq \|\nabla f(\mathbf{a})\|\|\mathbf{v}\| + \|\mathbf{v}\||E(\mathbf{a},\mathbf{v})|.$$

This shows that $f(\mathbf{a}+\mathbf{v}) \to f(\mathbf{a})$ as $\|\mathbf{v}\| \to 0$, so f is continuous at $\mathbf{a}$.

Example

Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by,

$$g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{, if } (x,y) \neq (0,0), \\ 0 & \text{, if } (x,y) = (0,0). \end{cases}$$

(i) Using the definition, show that $\frac{\partial}{\partial x} g(0,0) = 0$ and $\frac{\partial}{\partial y} g(0,0) = 0$.

(ii) Check the continuity of g at (0, 0).

(iii) Check the differentiability of g at (0, 0).

(iv) What conclusions can be obtained from above results on the differentiability of scalar fields and their partial derivatives at some points?

(i)

$$\begin{aligned}
\frac{\partial}{\partial x} g(x,y) &= \lim_{h\to 0} \frac{g(x+h,y)-g(x,y)}{h} \\
\frac{\partial}{\partial x} g(0,0) &= \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{h.0}{h^2+0} - 0\right) \\
&= 0.
\end{aligned}$$

Example
Solution

$$
\begin{aligned}
\frac{\partial}{\partial y} g(x,y) &= \lim_{h\to 0} \frac{g(x,y+h)-g(x,y)}{h} \\
\frac{\partial}{\partial y} g(0,0) &= \lim_{h\to 0} \frac{g(0,0+h)-g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{0.h}{0+h^2}-0\right) \\
&= 0.
\end{aligned}
$$

(ii) Consider the limit of the function $g(x,y)$ when $(x,y) \to (0,0)$ along the path $y = mx$, where $m \in \mathbb{R}$. Then we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} g(x,mx) &= \lim_{x\to 0} \frac{x.mx}{x^2+(mx)^2} \\ &= \frac{m}{1+m^2}. \end{aligned}$$

This limit changes when m changes. That is limit is not unique. Therefore $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exists. It implies that g is not continuous at $(0,0)$.

(iii) Since g is not continuous at $(0, 0)$, g is not differentiable at $(0, 0)$.

(iv) There exists some scalar fields which are not differentiable at a point but they have partial derivatives at that point.

The relationship between directional derivative and gradient vector

- When $\mathbf{y}$ is a unit vector the directional derivative $f'(\mathbf{a};\mathbf{y})$ has a simple geometric relation to the gradient vector.
- Assume that $\nabla f(\mathbf{a}) \neq \mathbf{0}$ and let θ denote the angle between $\mathbf{y}$ and $\nabla f(\mathbf{a})$.
- Then we have

$$f'(\mathbf{a};\mathbf{y}) = \nabla f(\mathbf{a}).\mathbf{y} = \|\nabla f(\mathbf{a})\|\|\mathbf{y}\| \cos\theta = \|\nabla f(\mathbf{a})\| \cos\theta.$$

The relationship between directional derivative and gradient vector
Cont...

- This shows that the directional derivative is simply the component of the gradient vector in the direction of $\mathbf{y}$.
- The derivative is largest when $\cos\theta = 1$, that is, when $\mathbf{y}$ has the same direction as $\nabla f(\mathbf{a})$.

The relationship between directional derivative and gradient vector
Cont...

- In other words, at a given point $\mathbf{a}$, the scalar field undergoes its maximum rate of change in the direction of the gradient vector.
- Moreover, this maximum is equal to the length of the gradient vector.
- When $\nabla f(\mathbf{a})$ is orthogonal to $\mathbf{y}$, the directional derivative $f'(\mathbf{a};\mathbf{y})$ is 0.

A sufficient condition for differentiability

- If f is differentiable at $\mathbf{a}$, then all partial derivatives $D_1 f(\mathbf{a}), ..., D_n f(\mathbf{a})$ exist.
- However, the existence of all these partials does not necessarily imply that f is differentiable at $\mathbf{a}$.

- A counter example is provided by the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

  discussed earlier.

- For this function, both partial derivatives $D_1 f(\mathbf{0})$ and $D_2 f(\mathbf{0})$ exist.

- But f is not continuous at $\mathbf{0}$, hence f cannot be differentiable at $\mathbf{0}$.

Theorem (7.2)
A sufficient condition for differentiability

Assume that the partial derivatives $D_1 f, ..., D_n f$ exist in some n-ball $B(\mathbf{a})$ and are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$.

- The above theorem shows that the existence of continuous partial derivatives at a point implies differentiability at that point.
- A scalar field satisfying the hypothesis of Theorem (7.2) is said to be continuously differentiable at $\mathbf{a}$.

Example

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function defined such that

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & ; \text{if } x \neq 0, \\ 0 & ; \text{if } x = 0. \end{cases}$$

(a) Evaluate $f_x(x,y)$ and $f_y(x,y)$.

(b) Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(x,y) = (0,0)$.

(c) Show that f is not continuous at $\mathbf{0}$.

(d) Using this result what can you say about differentiability of f at the point $(0,0)$?

(a)

$$
\begin{aligned}
f_x(x,y) &= \frac{(x^2+y^4)y^2 - xy^2(2x)}{(x^2+y^4)^2} \\
&= \frac{y^6 - x^2y^2}{(x^2+y^4)^2} \\
f_y(x,y) &= \frac{(x^2+y^4)2xy - xy^2 4y^3}{(x^2+y^4)^2} \\
&= \frac{2x^3y - 2xy^5}{(x^2+y^4)^2}.
\end{aligned}
$$

(b) Consider the limit of $f_x(x,y)$ when $(x,y) \to (0,0)$ along the path $y = mx$, where $m \in \mathbb{R}$. Then we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f_x(x,y) &= \lim_{x\to 0} f_x(x,mx) \\ &= \lim_{x\to 0} \frac{(mx)^6 - x^2(mx)^2}{(x^2+(mx)^4)^2} \\ &= \lim_{x\to 0} \frac{mx^6 - m^2x^4}{x^4(1+m^4x^2)^2} \\ &= -m^2. \end{aligned}$$

This limit depends on m. That is limit is not unique. Therefore the limit of $f_x(x,y)$ when $(x,y) \to (0,0)$ does not exist. So, $f_x(x,y)$ is not continuous at $(x,y) = (0,0)$.

Example
Solution

Consider the limit of $f_x(x, y)$ when $(x, y) \to (0, 0)$ along the path $y = bx$, where $b \in \mathbb{R}$. Then we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f_y(x, y) &= \lim_{x\to 0} f_y(x, bx) \\
&= \lim_{x\to 0} \frac{2x^3(bx) - 2x(bx)^5}{(x^2 + (bx)^4)^2} \\
&= \lim_{x\to 0} \frac{2bx^4 - 2b^5x^6}{x^4(1 + n^4x^2)^2} \\
&= 2b.
\end{aligned}$$

This limit also depends on b. That is limit is not unique.
Therefore the limit of $f_y(x, y)$ when $(x, y) \to (0, 0)$ does not exist.
So, $f_y(x, y)$ is not continuous at $(x, y) = (0, 0)$.

Example
Solution

(c) Exercise.

(d) Since $f(x, y)$ is not continuous at $(0, 0)$, $f(x, y)$ is not differentiable at $(0, 0)$.

Thank you!