Real Analysis III
(MAT312β)

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Directional Derivatives and Partial Derivatives

Why do we need directional derivatives?

- Suppose $y = f(x)$. Then the derivative $f'(x)$ is the rate at which y changes when we let x vary.
- Since f is a function on the real line, so the variable can only increase or decrease along that single line.
- In one dimension, there is only one "direction" in which x can change.

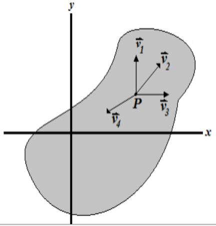
Why do we need directional derivatives?
Cont...

- Given a function of two or more variables like $z = f(x, y)$, there are infinitely many different directions from any point in which the function can change.
- We know that we can represent directions as vectors, particularly unit vectors when its only the direction and not the magnitude that concerns us.
- Directional derivatives are literally just derivatives or rates of change of a function in a particular direction.

Why do we need directional derivatives?
Cont...

- Let P is a point in the domain of $f(x, y)$ and vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$ represent possible directions in which we might want to know the rate of change of $f(x, y)$.

- Suppose we may want to know the rate at which $f(x, y)$ is changing along or in the direction of the vector, $\mathbf{v_3}$, which would be the direction along the x-axis.



Definition

If $\mathbf{y}$ is a unit vector, then

$$f'(\mathbf{a};\mathbf{y}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h},$$

the derivative $f'(\mathbf{a};\mathbf{y})$ is called the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{y}$.

Directional derivative of $f(x, y)$ at (a, b) in the direction of $\mathbf{u}$

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, we define the directional derivative $f_{\mathbf{u}}$ at the point (a, b) by

$$\begin{aligned} f_{\mathbf{u}}(a, b) &= \text{Rate of change of } f(x, y) \text{ in the direction of } \mathbf{u} \\ &\quad \text{at the point } (a, b) \\ &= \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \end{aligned}$$

provided that the limit exists.

Example 1

Compute the directional derivative of $f(x, y) = x + y^2$ at the point (4, 0) in the direction $\mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$.

Example 1
Solution

The norm of $\mathbf{u}$, that is $\|\mathbf{u}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$. Thus $\mathbf{u}$ is a unit vector.

$$
\begin{aligned}
f_{\mathbf{u}}(4,0) &= \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h\frac{1}{2}, 0+h\frac{\sqrt{3}}{2}) - f(4,0)}{h} \\
&= \lim_{h\to 0} \frac{(4+h\frac{1}{2}) + \left(h\frac{\sqrt{3}}{2}\right)^2 - 4}{h} \\
&= \lim_{h\to 0} \frac{4+\frac{h}{2}+\frac{3h^2}{4} - 4}{h} = \lim_{h\to 0}\left(\frac{1}{2}+\frac{3}{4}h\right) = \frac{1}{2}
\end{aligned}
$$

Example 2

A scalar field f is defined on $\mathbb{R}^n$ by the equation $f(\mathbf{x}) = \mathbf{a}.\mathbf{x}$, where $\mathbf{a}$ is a constant vector. Compute $f'(\mathbf{x}; \mathbf{y})$ for arbitary $\mathbf{x}$ and $\mathbf{y}$.

Example 2
Solution

According to the definition, we have

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h}\\
f'(\mathbf{x};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{y})-f(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{\mathbf{a}.(\mathbf{x}+h\mathbf{y})-\mathbf{a}.\mathbf{x}}{h}\\
&= \lim_{h\to 0}\frac{h(\mathbf{a}.\mathbf{y})}{h}\\
&= \mathbf{a}.\mathbf{y}.
\end{aligned}
$$

Example 3

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a given linear transformation. Compute the derivative $f'(\mathbf{x}; \mathbf{y})$ for the scalar field defined on $\mathbb{R}^n$ by the equation $f(\mathbf{x}) = \mathbf{x}.T(\mathbf{x})$.

Example 3
Solution

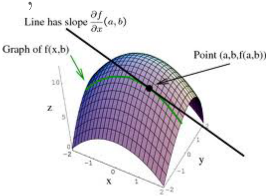
According to the definition, we have

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h}\\
f'(\mathbf{x};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{y})-f(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{(\mathbf{x}+h\mathbf{y}).T(\mathbf{x}+h\mathbf{y})-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{(\mathbf{x}+h\mathbf{y}).(T(\mathbf{x})+hT(\mathbf{y}))-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{\mathbf{x}.T(\mathbf{x})+h\mathbf{x}.T(\mathbf{y})+h\mathbf{y}.T(\mathbf{x})+h^2\mathbf{y}.T(\mathbf{y})-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{h\mathbf{x}.T(\mathbf{y})+h\mathbf{y}.T(\mathbf{x})+h^2\mathbf{y}.T(\mathbf{y})}{h}\\
&= \mathbf{x}.T(\mathbf{y})+\mathbf{y}.T(\mathbf{x}).
\end{aligned}
$$

Partial derivatives

- If $\mathbf{y}$ is a unit vector, the derivative $f'(\mathbf{a};\mathbf{y})$ is called the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{y}$.
- In particular, if $\mathbf{y} = \mathbf{e}_k$ (the k^{th} unit coordinate vector) the directional derivative $f'(\mathbf{a};\mathbf{e}_k)$ is called partial derivative with respect to $\mathbf{e}_k$ and is also denoted by the symbool $D_k f(\mathbf{a})$.
- Thus

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}, \\
f'(\mathbf{a};\mathbf{e}_k) &= D_k f(\mathbf{a}).
\end{aligned}
$$



The following notations are also used for the partial derivative $D_k f(\mathbf{a})$:

(i) $D_k f(a_1, ..., a_n)$,

(ii) $\frac{\partial f}{\partial x_k}(a_1, ..., a_n)$,

(iii) $f'_{x_k}(a_1, ..., a_n)$.

Sometimes the derivative f'_{x_k} is written without the prime as f_{x_k} or even more simply as f_k.

- In $\mathbb{R}^2$ the unit coordinate vectors are denoted by $\mathbf{i}$ and $\mathbf{j}$.
- If $\mathbf{a} = (a, b)$ the partial derivatives $f'(\mathbf{a}; \mathbf{i})$ and $f'(\mathbf{a}; \mathbf{j})$ are also written as

$$\frac{\partial f}{\partial x}(a, b) \text{ and } \frac{\partial f}{\partial y}(a, b),$$

respectively.

Partial derivatives in $\mathbb{R}^3$

Notations

- In $\mathbb{R}^3$ the unit coordinate vectors are denoted by **i**, **j**, and **k**.
- If $\mathbf{a} = (a, b, c)$ the partial derivatives $D_1 f(\mathbf{a})$, $D_2 f(\mathbf{a})$, and $D_3 f(\mathbf{a})$ are denoted by

$$\frac{\partial f}{\partial x}(a, b, c),\ \frac{\partial f}{\partial y}(a, b, c),\ \text{and}\ \frac{\partial f}{\partial z}(a, b, c),$$

respectively.

Partial derivatives of higher order

- Partial differentiation produces new scalar fileds $D_1 f, ..., D_n f$ from a given scalar field f.
- The partial derivatives $D_1 f, ..., D_n f$ are called first order partial derivatives of f.
- For function of two variables there are four second order partial derivatives, which are written as follows:

$$D_1(D_1 f) = \frac{\partial^2 f}{\partial x^2}, \qquad D_2(D_2 f) = \frac{\partial^2 f}{\partial y^2}$$

$$D_1(D_2 f) = \frac{\partial^2 f}{\partial x \partial y}, \qquad D_2(D_1 f) = \frac{\partial^2 f}{\partial y \partial x}.$$

Partial derivatives of higher order
Cont...

- In the above, $D_1(D_2 f)$ means the partial derivative of $(D_2 f)$ with respect to the first variable.
- We sometimes use the notation $D_{i,j} f$ for the second-order partial derivative $D_i(D_j f)$.
- For example, $D_{1,2} f = D_1(D_2 f)$.

Partial derivatives of higher order
Cont...

- In the ∂-notation we indicate the order of derivatives by writing

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

- This may or may not be equal to the other mixed partial derivative,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Partial derivatives of higher order
Remark

- We shall prove later that the two mixed partials $D_1(D_2 f)$ and $D_2(D_1 f)$ are equal at a point if one of them is continuous in a neighborhood of the point.

Example

Consider the function

$$f(x, y) = x^2 + 5xy - 4y^2.$$

Find the second order partial derivatives of f.

$$\frac{\partial f}{\partial x} = 2x + 5y \qquad \frac{\partial f}{\partial y} = 5x - 8y.$$

- A second order partial derivative should be a partial derivative of a first order partial derivative.
- So, first take two different first order partial derivatives, with respect to x or y and then, for each of those, you can take a partial derivative a second time with respect to x or y.

Example

Solution⇒Cont...

$$
\begin{aligned}
D_1(D_1 f) &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x + 5y) = 2, \\
D_2(D_1 f) &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x + 5y) = 5, \\
D_1(D_2 f) &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(5x - 8y) = 5, \\
D_2(D_2 f) &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x - 8y) = -8.
\end{aligned}
$$

Note that f_{xy} and f_{yx} are equal in this example. While this is not always the case.

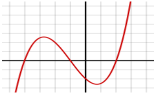
Directional derivatives and continuity
Differentiable function in one dimensional space

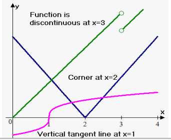
- If a is a point in the domain of a function f, then f is said to be differentiable at a if the derivative $f'(a)$ exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- In calculus, a differentiable function is a function whose derivative exists at each point in its domain.

- The graph of a differentiable function must be relatively smooth, and cannot contain any breaks, bends, cusps, or any points with a vertical tangent.

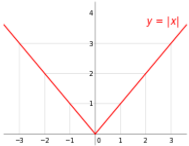




Directional derivatives and continuity

Differentiability and continuity in one dimensional space

- If f is differentiable at a point a, then f must also be continuous at a.
- In particular, any differentiable function must be continuous at every point in its domain.
- The converse does not hold: a continuous function need not be differentiable.
- For example, the absolute value function is continuous at $x = 0$ but it is not differentiable at $x = 0$.



- In one-dimensional space, the existence of the derivative of a function f at a point implies continuity at that point.
- This can easily be shown by considering the definition of the derivative of a single variable function.

$$\begin{aligned} f(a+h)-f(a) &= \frac{f(a+h)-f(a)}{h}.h \\ \lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.h \end{aligned}$$

$$
\begin{aligned}
\lim_{h\to 0}(f(a+h)-f(a)) &= f'(a).0 \\
\lim_{h\to 0}(f(a+h)-f(a)) &= 0 \\
\lim_{h\to 0} f(a+h)-\lim_{h\to 0} f(a) &= 0 \\
\lim_{h\to 0} f(a+h) &= \lim_{h\to 0} f(a) \\
\lim_{h\to 0} f(a+h) &= f(a).
\end{aligned}
$$

- This shows that the existence of $f'(a)$ implies continuity of f at a.

Assume the derivative $f'(\mathbf{a};\mathbf{y})$ exists for some $\mathbf{y}$. Then if $h \neq 0$ we can write

$$\begin{aligned}
f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a}) &= \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}.h \\
\lim_{h\to 0}\left(f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})\right) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}.h \\
\lim_{h\to 0}\left(f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})\right) &= f'(\mathbf{a};\mathbf{y}).0 \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) - \lim_{h\to 0} f(\mathbf{a}) &= 0 \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) &= \lim_{h\to 0} f(\mathbf{a}) \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) &= f(\mathbf{a}).
\end{aligned}$$

- This means that $f(\mathbf{x}) \to f(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$ along a straight line through $\mathbf{a}$ having direction $\mathbf{y}$.
- If $f'(\mathbf{a}; \mathbf{y})$ exists for every vector $\mathbf{y}$, then $f(\mathbf{x}) \to f(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$ along every line through $\mathbf{a}$.
- This seems to suggest that f is continuous at $\mathbf{a}$.
- Surprisingly enough, this conclusion need not be true.

Let f be the scalar field defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the above scalar field has directional derivative in every direction at $\mathbf{0}$ but which is not continuous at $\mathbf{0}$.

Let $\mathbf{a} = (0,0)$ and let $\mathbf{y} = (a,b)$ be any vector. If $a \neq 0$ and $h \neq 0$ we have

$$\begin{aligned}
\frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h} &= \frac{f(\mathbf{0}+h\mathbf{y}) - f(\mathbf{0})}{h} \\
&= \frac{f(h\mathbf{y}) - f(\mathbf{0})}{h} \\
&= \frac{f(h(a,b)) - f(0,0)}{h} \\
&= \frac{f(h(a,b))}{h} \\
&= \frac{f(ha,hb)}{h} \\
&= \frac{1}{h}\left(\frac{(ha)(hb)^2}{(ha)^2 + (hb)^4}\right) = \frac{ab^2}{a^2 + h^2 b^4}.
\end{aligned}$$

Directional derivatives and continuity

Example⇒Cont...

$$\begin{aligned}
\lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} &= \lim_{h\to 0}\frac{ab^2}{a^2+h^2b^4} \\
\lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} &= \frac{ab^2}{a^2+0.b^4}. \\
f'(\mathbf{0};\mathbf{y}) &= \frac{b^2}{a}.
\end{aligned}$$

- If $\mathbf{y} = (0, b)$ we find, in a similar way, that $f'(\mathbf{0}; \mathbf{y}) = 0$.
- Therefore $f'(\mathbf{0}; \mathbf{y})$ exists for all directions $\mathbf{y}$.
- Also, $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$ along any straight line through the origin.
- However, at each point of the parabola $x = y^2$ (except at the origin) the function f has the value $1/2$.

- Since such points exist arbitrarily close to the origin and since $f(\mathbf{0}) = 0$, the function f is not continuous at $\mathbf{0}$.
- The above example describes a scalar field which has a directional derivative in every direction at $\mathbf{0}$ but which is not continuous at $\mathbf{0}$.

- The above example shows that the existence of all directional derivatives at a point fails to imply continuity at that point.
- For this reason, directional derivatives are somewhat unsatisfactory extension of the one-dimensional concept of derivative.
- A more suitable generalization exists which implies continuity and, at the same time, permits us to extend the principal theorems of one dimensional derivative theory to the higher demensional case.
- This is called the **total derivative**.

Thank you!