Real Analysis III

(MAT312β)

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The Derivative of a Scalar Field with Respect to a Vector

Motivative example

- Suppose a person is at point $\mathbf{a}$ in a heated room with an open window.
- Let $f(\mathbf{a})$ is the temperature at a point $\mathbf{a}$.
- If the person moves toward the window temperature will decrease, but if the person moves toward heater it will increase.
- In general, the manner in which a field changes will depend on the direction in which we move away from $\mathbf{a}$.

Motivative example
Cont...

- Let $f : \mathbb{S} \to \mathbb{R}$ be a scalar field where $\mathbb{S} \subseteq \mathbb{R}^n$ and let $\mathbf{a}$ be an interior point of $\mathbb{S}$.
- We are going to study about how the field changes as we move from $\mathbf{a}$ to a nearby point.

Motivative example
Cont...

- Suppose moving direction is given by the vector $\mathbf{y}$.
- That is suppose we move from $\mathbf{a}$ toward another point $\mathbf{a}+\mathbf{y}$ along the line segment joining $\mathbf{a}$ and $\mathbf{a}+\mathbf{y}$.
- Each point on this segment has the form $\mathbf{a}+h\mathbf{y}$, where h is a real number.
- The distance from $\mathbf{a}$ to $\mathbf{a}+h\mathbf{y}$ is $\|h\mathbf{y}\| = |h|\|\mathbf{y}\|$.

Motivative example
Cont...

- Since $\mathbf{a}$ is an interior point of $\mathbb{S}$, there is an n-ball $\mathbf{B}(\mathbf{a};\mathbf{r})$ lying entirely in $\mathbb{S}$.
- If h is chosen so that $|h|\|\mathbf{y}\| < r$, the segment from $\mathbf{a}$ to $\mathbf{a}+h\mathbf{y}$ will lie in $\mathbb{S}$.
- We keep $h \neq 0$ but small enough to guarantee that $\mathbf{a}+h\mathbf{y} \in \mathbb{S}$.
- So, then from the difference quotient we have,

$$\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h}.$$

Motivative example

Cont...

- If we consider the above quotient, the numerator tells us how much the function changes when we move from $\mathbf{a}$ to $\mathbf{a} + h\mathbf{y}$.
- The quoteint itself is called the **average rate of change** of f over the line segmengnt joining $\mathbf{a}$ to $\mathbf{a} + h\mathbf{y}$.
- We are interested in the behavior of this quotient as $h \to 0$.
- This leads us to the following definition.

Definition

The derivative of a scalar field with respect to a vector

Given a scalar field $f : \mathbb{S} \to \mathbb{R}$, where $\mathbb{S} \subseteq \mathbb{R}^n$. Let $\mathbf{a}$ be an interior point of $\mathbb{S}$ and let $\mathbf{y}$ be an arbitrary point in $\mathbb{R}^n$. The derivative of f at $\mathbf{a}$ with respect to $\mathbf{y}$ is denoted by the symbol $f'(\mathbf{a};\mathbf{y})$ and is defined by the equation

$$f'(\mathbf{a};\mathbf{y}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}, \tag{1}$$

when the limit on the right exists.

Example 1

If $\mathbf{y} = \mathbf{0}$, the difference quotient (1) is 0 for every $h \neq 0$, so $f'(\mathbf{a}; \mathbf{0})$ always exists and equals 0.

$$\begin{aligned}
f'(\mathbf{a}; \mathbf{y}) &= \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a}; \mathbf{0}) &= \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{0}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a}; \mathbf{0}) &= \lim_{h \to 0} \frac{f(\mathbf{a}) - f(\mathbf{a})}{h} \\
&= 0.
\end{aligned}$$

Example 2
Derivative of a linear transformation

If $f : \mathbb{S} \to \mathbb{R}$ is a linear transformation, then $f(\mathbf{a} + h\mathbf{y}) = f(\mathbf{a}) + hf(\mathbf{y})$. From the definition we have,

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{f(\mathbf{a}) + hf(\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{hf(\mathbf{y})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= f(\mathbf{y}).
\end{aligned}
$$

Therefore, the derivative of linear transformation with respect to $\mathbf{y}$ is equal to the value of the function at $\mathbf{y}$.

Pre-requisite for theorem 4.1

To study how f behaves on the line passing through $\mathbf{a}$ and $\mathbf{a} + \mathbf{y}$ for $\mathbf{y} \neq \mathbf{0}$ we introduce the function

$$g(t) \;=\; f(\mathbf{a} + t\mathbf{y}).$$

The next theorem relates the derivatives $g'(t)$ and $f'(\mathbf{a} + t\mathbf{y}; \mathbf{y})$.

Theorem 4.1

Let $g(t) = f(\mathbf{a} + t\mathbf{y})$. If one of the derivatives $g'(t)$ or $f'(\mathbf{a} + t\mathbf{y}; \mathbf{y})$ exists then the other also exists and the two are equal,

$$g'(t) \quad = \quad f'(\mathbf{a} + t\mathbf{y}; \mathbf{y}).$$

In particular, when $t = 0$ we have $g'(0) = f'(\mathbf{a}; \mathbf{y})$.

Forming the difference quotient for g, we have,

$$
\begin{aligned}
\frac{g(t+h)-g(t)}{h} &= \frac{f(\mathbf{a}+(t+h)\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
\frac{g(t+h)-g(t)}{h} &= \frac{f(\mathbf{a}+t\mathbf{y}+h\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} &= \lim_{h\to 0}\frac{f(\mathbf{a}+t\mathbf{y}+h\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
g'(t) &= f'(\mathbf{a}+t\mathbf{y};\mathbf{y}).
\end{aligned}
$$

Example

Compute $f'(\mathbf{a}; \mathbf{y})$ if $f(\mathbf{x}) = \|\mathbf{x}\|^2$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Example
Solution

We let

$$\begin{aligned}
g(t) &= f(\mathbf{a}+t\mathbf{y}) \\
&= \|\mathbf{a}+t\mathbf{y}\|^2 \text{ since } f(\mathbf{x}) = \|\mathbf{x}\|^2 \\
&= (\mathbf{a}+t\mathbf{y}).(\mathbf{a}+t\mathbf{y}) \text{ since } \|\mathbf{x}\|^2 = \mathbf{x}.\mathbf{x} \\
&= \mathbf{a}.\mathbf{a} + t\mathbf{a}.\mathbf{y} + t\mathbf{y}.\mathbf{a} + t^2\mathbf{y}.\mathbf{y} \\
g(t) &= \mathbf{a}.\mathbf{a} + 2t\mathbf{a}.\mathbf{y} + t^2\mathbf{y}.\mathbf{y} \\
g'(t) &= 0 + 2\mathbf{a}.\mathbf{y} + 2t\mathbf{y}.\mathbf{y}
\end{aligned}$$

We need to find $f'(\mathbf{a};\mathbf{y})$. If we subsitute

$$\begin{aligned}
f'(\mathbf{a}+0\mathbf{y};\mathbf{y}) &= g'(0) = 2\mathbf{a}.\mathbf{y} \\
f'(\mathbf{a};\mathbf{y}) &= 2\mathbf{a}.\mathbf{y}.
\end{aligned}$$

Thank you!