Real Analysis III
(MAT312β)

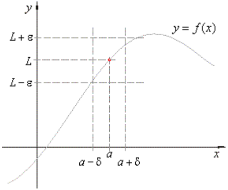
Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Limits and Continuity

Limits in one dimensional space

- When we write $\lim_{x\to a} f(x) = L$, we mean that f can be made as close as we want to L, by taking x close enough to a but not equal to a.
- In here the function f has to be defined near a, but not necessarily at a.
- The purpose of limit is to determine the behavior of $f(x)$ as x gets closer to a.

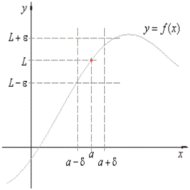


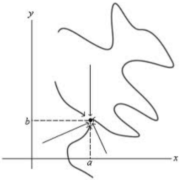
Limits in higher dimensional spaces

- The domain of functions of two variables is a subset of $\mathbb{R}^2$, in other words it is a set of pairs.
- A point in R^2 is of the form (x, y).
- So, the equivalent of $\lim_{x\to a} f(x)$ will be $\lim_{(x,y)\to(a,b)} f(x, y)$.
- For functions of three variables, the equivalent of $\lim_{x\to a} f(x)$ will be $\lim_{(x,y,z)\to(a,b,c)} f(x, y, z)$, and so on.

Difficulty of getting limits in higher dimensional spaces

- While x could only approach a from two directions, from the left or from the right, (x, y) can approach (a, b) from infinitely many directions.
- In fact, it does not even have to approach (a, b) along a straight path as shown in figure.





Difficulty of getting limits in higher dimensional spaces
Cont...

- With functions of one variable, one way to show a limit existed, was to show that the limit from both directions existed and were equal.
- That is $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.
- Equivalently, when the limits from the two directions were not equal, we concluded that the limit did not exist.

- For functions of several variables, we would have to show that the limit along every possible path exist and are the same.
- The problem is that there are infinitely many such paths.
- To show a limit does not exist, it is still enough to find two paths along which the limits are not equal.
- In view of the number of possible paths, it is not always easy to know which paths to try.

Example 1

Find the limit

$$\lim_{(x,y)\to(2,3)} \frac{3x^2y}{x^2+y^2}.$$

Notice that the point $(2, 3)$ does not cause division by zero or other domain issues. So,

$$\lim_{(x,y)\to(2,3)} \frac{3x^2y}{x^2+y^2} = \frac{3(2)^2(3)}{(2)^2+(3)^2} = \frac{36}{13}.$$

Example 2

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}.$$

Example 2
Solution

$$\begin{aligned}
\text{Let } x = 0 \Rightarrow \quad &= \frac{x^2}{x^2+y^2} \\
&= \frac{0}{0+y^2} \\
&= 0 \\
\text{Let } y = 0 \Rightarrow \quad &= \frac{x^2}{x^2+y^2} \\
&= \frac{x^2}{x^2+0} \\
&= \frac{x^2}{x^2} \\
&= 1
\end{aligned}$$

Since we got two different results, the limit does not exist.

Example 3

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 - y^2}{x^2 + y^2}.$$

Example 3
Solution

$$\begin{aligned} \text{Let } x = 0 \Rightarrow \quad &= \frac{3x^2 - y^2}{x^2 + y^2} \\ &= \frac{-y^2}{y^2} \\ &= -1 \\ \text{Let } y = 0 \Rightarrow \quad &= \frac{3x^2 - y^2}{x^2 + y^2} \\ &= \frac{3x^2}{x^2} \\ &= 3 \end{aligned}$$

Again, the limit does not exist.

Example 4

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

Example 4
Solution

$$\begin{aligned}
\text{Let } x = 0 \Rightarrow \quad &= \frac{xy}{x^2 + y^2} \\
&= \frac{0y}{0 + y^2} = 0 \\
\text{Let } y = 0 \Rightarrow \quad &= \frac{xy}{x^2 + y^2} \\
&= \frac{x0}{x^2 + 0} = 0
\end{aligned}$$

We approached $(0, 0)$ from two different directions and got the same result, but it doesnt necessarily mean $f(x, y)$ has that limit.

Let $(x, y) \to (0, 0)$ along the line $y = mx$. Then

$$\begin{aligned}\frac{xy}{x^2 + y^2} &= \frac{x(mx)}{x^2 + (mx)^2} \\ &= \frac{mx^2}{x^2 + m^2x^2} \\ &= \frac{m}{1 + m^2}\end{aligned}$$

This shows that the limit depends on the choice of m. Therefore, the limit does not exist.

Remark

- If we approach (a, b) from two different directions and get two different results, then $f(x, y)$ does not have a limit.
- If we approach (a, b) from two different directions and get the same result, then it doesnt necessarily mean $f(x, y)$ has that limit.
- We have to get the same limit no matter from which direction we approach (a, b).
- To do this, we would sometimes have to use the definition of the limit of a function of two variables in order to ensure that we have the correct limit.

The $\epsilon\delta$-definition of limit

The function $f(x, y)$ has the limit L as $(x, y) \to (a, b)$ provided that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

To convert basic definition of $\lim_{(x,y)\to(a,b)} f(x,y) = L$ into above formal definition, we replace the phrase,

- "$f(x,y)$ is arbitrarily close to L" with "$|f(x,y) - L| < \epsilon$ for an arbitrarily small positive number ϵ," and,
- "for all $(x,y) \neq (a,b)$ sufficiently close to (a,b)" with "for all (x,y) with $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ for a sufficiently small positive number δ."

The $\epsilon\delta$-definition of limit
Cont...

When we use the definition of a limit to show that a particular limit exists, we usually employ certain key or basic inequalities such as:

1. $|x| < \sqrt{x^2 + y^2}$
2. $\dfrac{x}{x+1} < 1$
3. $|y| < \sqrt{x^2 + y^2}$
4. $\dfrac{x^2}{x^2 + y^2} < 1$
5. $|x - a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-a)^2}$

Example 5

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}.$$

Example 5
Solution

$$
\begin{aligned}
\text{Let } x = 0 \Rightarrow \quad &= \frac{x^2y}{x^2+y^2} \\
&= \frac{0}{0+y^2} = 0 \\
\text{Let } y = 0 \Rightarrow \quad &= \frac{x^2y}{x^2+y^2} \\
&= \frac{0}{x^2} = 0
\end{aligned}
$$

We suspect that the limit might be zero. Lets try the definition with $L = 0$.

Example 5

Solution⇒Cont...

$$|f(x,y) - L| < \epsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

$$\begin{aligned} |f(x,y) - 0| &= \left|\frac{x^2 y}{x^2 + y^2} - 0\right| \\ &= \left|\frac{x^2 y}{x^2 + y^2}\right| \\ &= |y|\left|\frac{x^2}{x^2 + y^2}\right| \end{aligned}$$

Example 5
Solution⇒Cont...

Now, since $\mid \dfrac{x^2}{x^2+y^2} \mid < 1$ then $\mid y \mid \mid \dfrac{x^2}{x^2+y^2} \mid < \mid y \mid$. So we then have

$$\mid y \mid \mid \frac{x^2}{x^2+y^2} \mid < \mid y \mid \quad < \quad \sqrt{x^2+y^2} = \sqrt{(x-0)^2+(y-0)^2} < \delta$$

Therefore, if $\delta = \epsilon$, the definition shows the limit does equal zero.

Continuity of a function

We consider a function $\mathbf{f} : \mathbb{S} \to \mathbb{R}^m$, where $\mathbb{S}$ is a subset of $\mathbb{R}^n$.
A function $\mathbf{f}$ is said to be continuous at $\mathbf{a}$ if $\mathbf{f}$ is defined at $\mathbf{a}$ and if

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

We say that $\mathbf{f}$ is continuous on a set $\mathbb{S}$ if $\mathbf{f}$ is continuous at each point of $\mathbb{S}$.

Continuity of a function

- Many familiar properties of limits and continuity of function of one variable can also be extended for function of several variables.
- For example, the usual theorems concerning limits and continuity of sums, products, and quotients also hold for scalar fields.
- For vector fields, quotients are not defined but we have the following theorem concerning sums, multification by scalars, inner products, and norms.

Theorem (3.1)

If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{c}$, then we also have:

(a) $\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})] = \mathbf{b} + \mathbf{c}$.

(b) $\lim_{\mathbf{x}\to\mathbf{a}} \lambda\mathbf{f}(\mathbf{x}) = \lambda\mathbf{b}$ for every scalar λ.

(c) $\lim_{\mathbf{x}\to\mathbf{a}}[\mathbf{f}(\mathbf{x}).\mathbf{g}(\mathbf{x})] = \mathbf{b}.\mathbf{c}$.

(d) $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{f}(\mathbf{x})\| = \|\mathbf{b}\|$.

Continuity and components of a vector field

If a vector field $\mathbf{f}$ has values in $\mathbb{R}^m$, each function value $\mathbf{f}(\mathbf{x})$ has m components and we can write

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_m(\mathbf{x})).$$

The m scalar fields $f_1, f_2, ..., f_m$ are called components of the vector field $\mathbf{f}$.

We shall prove that $\mathbf{f}$ is continuous at a point $\mathbf{a}$ if, and only if, each components f_k is continuous at that point.

$$\mathbf{f} \text{ is cts at } \mathbf{a} \in \mathbb{R}^n \Longleftrightarrow \text{Each } f_k(k = 1, 2, ..., m) \text{ is cts at } \mathbf{a} \in \mathbb{R}^n$$

Continuity and components of a vector field

Proof

Let $\mathbf{e}_k$ denote the k^{th} unit coordinate vector.

All the components of $\mathbf{e}_k$ are 0 except the k^{th}, which is equal to 1.

That is $\mathbf{e}_k = (0, 0, ..., 0, 1, 0, ..., 0)$. Then $f_k(\mathbf{x})$ is given by the dot product and

$$
\begin{aligned}
f_k(\mathbf{x}) &= \mathbf{f}(\mathbf{x}).\mathbf{e}_k \\
\lim_{\mathbf{x}\to\mathbf{a}} f_k(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}).\mathbf{e}_k \\
&= \mathbf{f}(\mathbf{a}).\mathbf{e}_k \\
&= (f_1(\mathbf{a}), f_2(\mathbf{a}), ..., f_k(\mathbf{a}), ..., f_m(\mathbf{a})).(0, 0, ..., 0, 1, 0, ..., 0) \\
&= f_k(\mathbf{a})
\end{aligned}
$$

Therefore, it implies that each point of continuity of $\mathbf{f}$ is also a point of continuity of f_k.

Moreover, since we have

$$\mathbf{f}(\mathbf{x}) = \sum_{k=1}^{m} f_k(\mathbf{x})\mathbf{e}_k,$$

repeated application of parts (a) and (b) of above theorem shows that a point of continuity of all m components $f_1, \ldots, f_m$ is also a point of continuity of $\mathbf{f}$.

Continuity of the identity function

The identity function $\mathbf{f}(\mathbf{x}) = \mathbf{x}$, is continuous everywhere in $\mathbb{R}^n$. Therefore its components are also continuous everywhere in $\mathbb{R}^n$. These are the n scalar fields given by

$$f_1(\mathbf{x}) = x_1,\ f_2(\mathbf{x}) = x_2, ..., f_n(\mathbf{x}) = x_n.$$

Continuity of the linear transformations

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We will prove that $\mathbf{f}$ is continuous at each point $\mathbf{a}$ in $\mathbb{R}^n$.

For the continuity of $\mathbf{f}$, we must show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{a}+\mathbf{h}) = \mathbf{f}(\mathbf{a}).$$

By linearity we have

$$\mathbf{f}(\mathbf{a}+\mathbf{h}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{h}).$$

It is suffices to show that

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{h}) = \mathbf{0}.$$

Let us write

$$\mathbf{h} = h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + ... + h_n\mathbf{e}_n.$$

Using linearity again we find that

$$\mathbf{f}(\mathbf{h}) = h_1\mathbf{f}(\mathbf{e}_1) + h_2\mathbf{f}(\mathbf{e}_2) + ... + h_n\mathbf{f}(\mathbf{e}_n).$$

Then by taking limit from both side we have,

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{h}) = \lim_{\mathbf{h}\to\mathbf{0}} \sum_{i=1}^{n} h_i\mathbf{f}(\mathbf{e}_i)$$

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{h}) = \sum_{i=1}^{n} \lim_{\mathbf{h}\to\mathbf{0}} h_i\mathbf{f}(\mathbf{e}_i).$$

Continuity of the linear transformations

Proof⇒Cont...

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{h}) = \sum_{i=1}^{n} \mathbf{f}(\mathbf{e}_i) \lim_{\mathbf{h}\to\mathbf{0}} h_i$$

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{h}) = \sum_{i=1}^{n} \mathbf{f}(\mathbf{e}_i) 0 = \mathbf{0}.$$

Theorem (3.2)

Let $\mathbf{f}$ and $\mathbf{g}$ be functions such that the composite function $(\mathbf{f} \circ \mathbf{g})$ is defined at $\mathbf{a}$, where

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}[\mathbf{g}(\mathbf{x})].$$

If $\mathbf{g}$ is continuous at $\mathbf{a}$ and if $\mathbf{f}$ is continuous at $\mathbf{g}(\mathbf{a})$, then the composition $(\mathbf{f} \circ \mathbf{g})$ is continuous at $\mathbf{a}$.

Theorem (3.2)
Proof

Let $\mathbf{y} = \mathbf{g}(\mathbf{x})$ and $\mathbf{b} = \mathbf{g}(\mathbf{a})$. Then we have

$$\mathbf{f}[\mathbf{g}(\mathbf{x})] - \mathbf{f}[\mathbf{g}(\mathbf{a})] = \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{b}).$$

By hypothesis, $\mathbf{y} \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$, so we have

$$\begin{aligned}
\lim_{\|\mathbf{x}-\mathbf{a}\|\to\mathbf{0}} \|\mathbf{f}[\mathbf{g}(\mathbf{x})] - \mathbf{f}[\mathbf{g}(\mathbf{a})]\| &= \lim_{\|\mathbf{y}-\mathbf{b}\|\to\mathbf{0}} \|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{b})\| \\
\lim_{\|\mathbf{x}-\mathbf{a}\|\to\mathbf{0}} \|\mathbf{f}[\mathbf{g}(\mathbf{x})] - \mathbf{f}[\mathbf{g}(\mathbf{a})]\| &= 0 \\
\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}[\mathbf{g}(\mathbf{x})] &= \mathbf{f}[\mathbf{g}(\mathbf{a})].
\end{aligned}$$

So, $(\mathbf{f} \circ \mathbf{g})$ is continuous at $\mathbf{a}$.

Example 1

Discuss the continuity of following functions.

1. $\sin(x^2y)$
2. $\log(x^2 + y^2)$
3. $\dfrac{e^{x+y}}{x + y}$
4. $\log[\cos(x^2 + y^2)]$

These examples are continuous at all points at which the functions are defined.

1. The first is continuous at all points in the plane.
2. The second at all points except the origin.
3. The third is continuous at all points (x, y) at which $x + y \neq 0$.
4. The fourth at all points at which $x^2 + y^2$ is not an odd multiple of $\pi/2$. The set of (x, y) such that $x^2 + y^2 = n\pi/2$, $n = 1, 3, 5, ...$, is a family of circles centered at the origin.

Remark 1

These examples show that the discontinuities of a function of two variables may consist of isolated points, entire curves, or families of curves.

Remark 2

A function of two variables may be continuous in each variable separately and yet be discontinuous as a function of the two variables together.

Example

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0),\ f(0,0) = 0.$$

- For points (x, y) on the x-axis we have $y = 0$ and $f(x,y) = f(x,0) = 0$, so the function has constant value 0 everywhere on the x-axis.
- Therefore, if we put $y = 0$ and think of f as a function of x alone, f is continuous at $x = 0$.
- Similarly, f has constant value 0 at all points on y-axis, so if we put $x = 0$ and think of f as a function of y alone, f is continuous at $y = 0$.

Example
Cont...

- However, as a function of two variables, f is not continuous at the origin.
- In fact, at each point of the line $y = x$ (except the origin) the function has the constant value $1/2$ because $f(x, x) = x^2/(2x^2) = 1/2$.
- Since there are points on this line which are close to the origin and since $f(0, 0) \neq 1/2$, the function is not continuous at $(0, 0)$.

Remark 3

If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, and if the one dimensional limits

$$\lim_{x\to a} f(x,y) \text{ and } \lim_{y\to b} f(x,y)$$

both exists, prove that

$$\lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right) = \lim_{y\to b}\left(\lim_{x\to a} f(x,y)\right) = L.$$

The two limits in the above equation are called **iterated limits**; the example shows that the existence of two-dimensional limit and of the two one-dimensional limits implies the existence and equality of the two iterated limits. (The converse is not always true).

Remark 3
Cont...

Since $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we have

$$\Rightarrow \lim_{\|(x,y)-(a,b)\|\to 0} \|f(x,y) - L\| = 0$$
$$\Rightarrow \lim_{\sqrt{(x-a)^2+(y-b)^2}\to 0} \|f(x,y) - L\| = 0$$
$$\Rightarrow \lim_{x\to a} \text{ and } \lim_{y\to b} \|f(x,y) - L\| = 0 \to \text{(A)}.$$

Remark 3
Cont...

Since one-dimensional limits, $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ both exists, let $\lim_{x\to a} f(x,y) = g(y)$ and $\lim_{y\to b} f(x,y) = h(x)$. Then we have,

$$\begin{aligned}\lim_{|x-a|\to 0} \|f(x,y) - g(y)\| &= 0 \to \text{(B)}\\ \lim_{|y-b|\to 0} \|f(x,y) - h(x)\| &= 0 \to \text{(C)}.\end{aligned}$$

Remark 3

Cont...

Now let us consider $\|h(x) - L\|$. Then

$$\begin{aligned} \|h(x) - L\| &= \|f(x,y) - f(x,y) + h(x) - L\| \\ \|h(x) - L\| &\leq \|f(x,y) - L\| + \|f(x,y) - h(x)\|. \end{aligned}$$

Letting $\lim_{x \to a}$ and $\lim_{y \to b}$ we have

$$\begin{aligned} \lim_{x \to a} \text{ and } \lim_{y \to b} \|h(x) - L\| &\leq \lim_{x \to a} \text{ and } \lim_{y \to b} \|f(x,y) - L\| + \\ &\quad \lim_{x \to a} \text{ and } \lim_{y \to b} \|f(x,y) - h(x)\| \\ &\leq 0 + 0 \text{ (From (A) and (C))}. \end{aligned}$$

Remark 3
Cont...

$$
\begin{aligned}
&\Rightarrow \lim_{x \to a} \text{ and } \lim_{y \to b} \|h(x) - L\| \leq 0 \\
&\Rightarrow \lim_{x \to a} \|h(x) - L\| \leq 0 \\
&\Rightarrow \lim_{x \to a} \|h(x) - L\| = 0 \\
&\Rightarrow \lim_{x \to a} h(x) = L \\
&\Rightarrow \lim_{x \to a} \left(\lim_{y \to b} f(x, y) \right) = L.
\end{aligned}
$$

Similarly we can show that

$$\lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right) = L.$$

Example

Let $f(x, y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$, whenever $x^2y^2 + (x-y)^2 \neq 0$.
Show that

$$\lim_{x \to 0}\left(\lim_{y \to 0} f(x, y)\right) = \lim_{y \to 0}\left(\lim_{x \to 0} f(x, y)\right) = 0,$$

but that $f(x, y)$ does not tend to a limit as $(x, y) \to (0, 0)$.
[**Hint:** Examine f on the line $y = x$.]

Example
Cont...

Consider the limit, $\lim_{y\to 0} f(x,y)$,

$$\begin{aligned}
\lim_{y\to 0} f(x,y) &= \lim_{y\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2} \\
&= 0 \\
\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) &= \lim_{x\to 0}(0) \\
&= 0 \to (1).
\end{aligned}$$

Example
Cont...

Consider the limit, $\lim_{x\to 0} f(x,y)$,

$$\begin{aligned}
\lim_{x\to 0} f(x,y) &= \lim_{x\to 0} \frac{x^2y^2}{x^2y^2+(x-y)^2} \\
&= 0 \\
\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) &= \lim_{y\to 0}(0) \\
&= 0 \to (2).
\end{aligned}$$

From (1) and (2) we have,

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0.$$

Example
Cont...

Let us consider the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ along the line $y = x$.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,x)\to(0,0)} f(x,x) \\ &= \lim_{x\to 0} \frac{x^4}{x^4+0} \\ &= 1. \end{aligned}$$

But we know that limit of a function should be unique. But above gives that the function has two limits 1 and 0. That is limit is not unique.
Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

Thank you!