Real Analysis III
(MAT312β)

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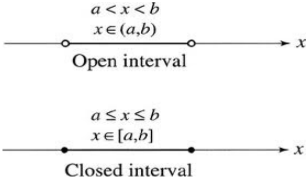
Open Sets

Open and closed intervals in $\mathbb{R}$

- Given two real numbers a and b with $a < b$, the closed interval $[a, b]$ is defined as the set of all real numbers x such that $a \leq x$ and $x \leq b$, or more concisely, $a \leq x \leq b$.
- The open interval (a, b) is defined as the set of all real numbers x such that $a < x < b$.
- The difference between the closed interval $[a, b]$ and the open interval (a, b) is that the end points a and b are elements of $[a, b]$ but are not elements of (a, b).



The importance of open intervals

- Open intervals play an important role in calculus of a function of a single real variable.
- Recall the definition of a local minimum (or maximum).
- Let $f : \mathbb{D} \to \mathbb{R}$, $(\mathbb{D} \subset \mathbb{R})$. Then we say that a point $c \in \mathbb{D}$ is a local minimum of f if there is an **open interval** U such that $c \in U$ and $f(x) \geq f(c)\ \forall x \in U \cap D$.
- Open intervals are used in many other definitions.

Generalization of open intervals into $\mathbb{R}^n$

- The concept of the open interval in $\mathbb{R}$ can be generalized to a subset of $\mathbb{R}^n$.
- The generalized version of the open interval is called as **open sets**.
- A number of important results that we shall obtain on the functions of several variables are only true when the domains of these functions are open sets.

Open balls

- Let $\mathbf{a} = (a_1, a_2, ..., a_n)$ be a given point in $\mathbb{R}^n$ and let r be a given positive number. The set of all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ such that,

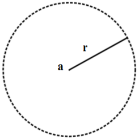
$$\|\mathbf{x} - \mathbf{a}\| < r,$$

  is called an open n-ball of radius r and center $\mathbf{a}$.

- We denote this set by $\mathbf{B}(\mathbf{a})$ or by $\mathbf{B}(\mathbf{a}; r)$.

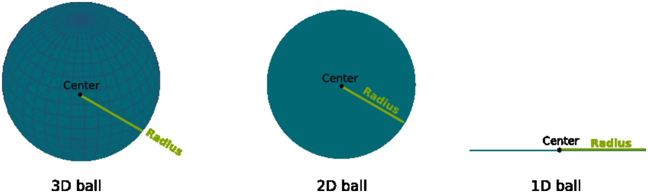
- The ball $\mathbf{B}(\mathbf{a}; r)$ consists of all points whose distance from $\mathbf{a}$ is less than r. So it can be written in symbolic form as,

$$\mathbf{B}(\mathbf{a}; r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\| < r\}.$$



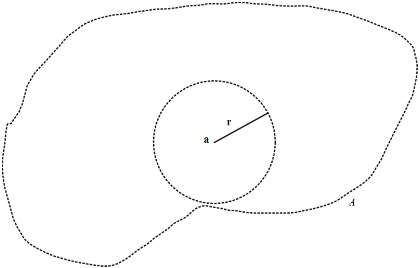
Open balls
Example

- In $\mathbb{R}^1$
  Open ball $\Rightarrow$ an open interval with center at a.
- In $\mathbb{R}^2$
  Open ball $\Rightarrow$ a circular disk with center at $\mathbf{a}$ and radius r.
- In $\mathbb{R}^3$
  Open ball $\Rightarrow$ a spherical solid with center at $\mathbf{a}$ and radius r.



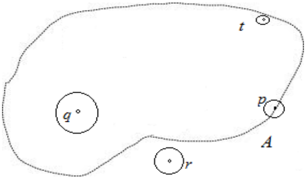
An interior point

Let A be a subset of $\mathbb{R}^n$, and assume that $\mathbf{a} \in A$. Then we say that $\mathbf{a}$ is an interior point of A if there is an open n-ball with center at $\mathbf{a}$, all of whose points belong to A.



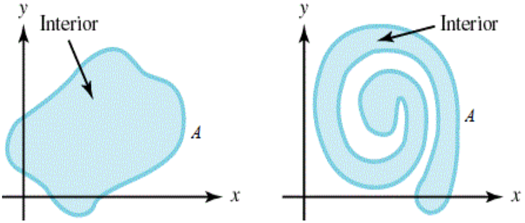
An interior point
Example

- Point t is an interior point.
- Point q is an interior point.
- Point r is not an interior point.
- Point p is not an interior point.



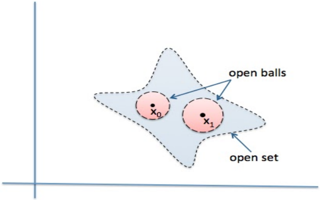
The interior of a set

Let A be a subset of $\mathbb{R}^n$. The set of all interior points of A is called the interior of A and it is denoted by *int* A or A^0.



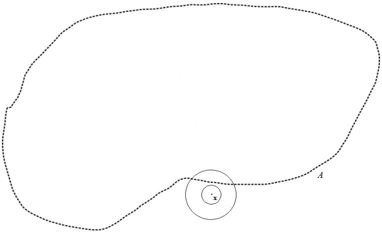
An open set

A set A in $\mathbb{R}^n$ is called open if all its points are interior points. In other words, A is open if and only if $A = int\ A$.



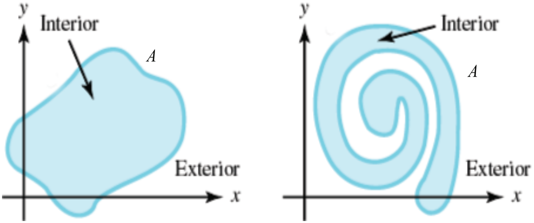
An exterior point

Let A be a subset of $\mathbb{R}^n$. A point $\mathbf{x}$ is said to be exterior to a set A in $\mathbb{R}^n$ if there is an n-ball $\mathbf{B}(\mathbf{x})$ containing no points of A.



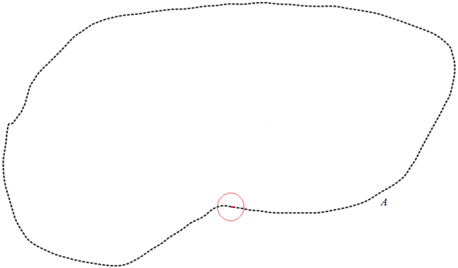
The exterior of a set

Let A be a subset of $\mathbb{R}^n$. The set of all points in $\mathbb{R}^n$ exterior to A is called the exterior of A and it is denoted by *ext* A.



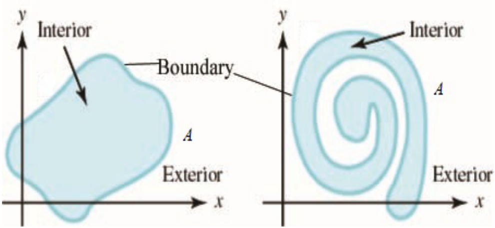
A boundary point

Let A be a subset of $\mathbb{R}^n$. A point which is neither exterior to A nor an interior point of A is called a boundary point of A.

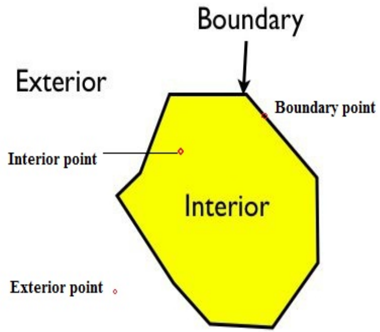


The boundary

Let A be a subset of $\mathbb{R}^n$. The set of all boundary points of A is called the boundary of A and it is denoted by ∂A.



Summary


Boundary
Exterior
Boundary point
Interior point
Interior
Exterior point

How to show a given set is an open set?

1. Any $\mathbf{x} \in \mathbb{S}$ is an interior point.
2. Any $\mathbf{x} \in \mathbb{S}$ is neither a boundary nor an exterior point.
3. $\mathbb{S}$ is open $\Leftrightarrow \mathbb{S}^c$ is closed.

Example

Let A_1 and A_2 are subset of $\mathbb{R}$ and both are open. Then show that the Cartesian product $A_1 \times A_2$ in $\mathbb{R}^2$ defined by,

$$A_1 \times A_2 = \{(a_1, a_2) | a_1 \in A_1 \text{ and } a_2 \in A_2\},$$

is also open.

Example
Solution

- To prove this, choose any point $\mathbf{a} = (a_1, a_2)$ in $A_1 \times A_2$.
- We must show that $\mathbf{a}$ is an interior point of $A_1 \times A_2$.
- A_1 is open in $\mathbb{R} \Rightarrow$ There is a 1-ball $\mathbf{B}(a_1; r_1)$ in A_1.
- A_2 is open in $\mathbb{R} \Rightarrow$ There is a 1-ball $\mathbf{B}(a_2; r_2)$ in A_2.
- Let $r = min\{r_1, r_2\}$.
- We can easily show that the 2-ball $\mathbf{B}(\mathbf{a}; r) \subseteq A_1 \times A_2$.

Example

Solution⇒Cont...

- In fact, if $\mathbf{x} = (x_1, x_2)$ is any point of $\mathbf{B}(\mathbf{a}; r)$ then $\|\mathbf{x} - \mathbf{a}\| < r$.
- So $|x_1 - a_1| < r_1 \Rightarrow \; x_1 \in \mathbf{B}(a_1; r_1)$.
- And $|x_2 - a_2| < r_2 \Rightarrow \; x_2 \in \mathbf{B}(a_2; r_2)$.
- Therefore $x_1 \in A_1$ and $x_2 \in A_2$, so $(x_1, x_2) \in A_1 \times A_2$.
- This proves that every point of $\mathbf{B}(\mathbf{a}; r)$ is in $A_1 \times A_2$.
- Therefore every point of $A_1 \times A_2$ is an interior point.
- So, $A_1 \times A_2$ is open.

Thank you !