Real Analysis III

(MAT312β)

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Sufficient Conditions for the Equality of Mixed Partial Derivatives

Mixed partial derivatives

- If f is a real-valued function of two variables, the two mixed partial derivatives $D_{1,2}f$ and $D_{2,1}f$ are not necessarily equal.
- By $D_{1,2}f$ we mean $D_1(D_2 f) = \partial^2 f/(\partial x \partial y)$, and by $D_{2,1}f$ we mean $D_2(D_1 f) = \partial^2 f/(\partial y \partial x)$.

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real valued function defined such that

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & ; \text{ if } (x, y) \neq (0, 0), \\ 0 & ; \text{ if } (x, y) = (0, 0). \end{cases}$$

Determine $D_{2,1}f(0, 0)$ and $D_{1,2}f(0, 0)$.

Example
Solution

The defintion of $D_{2,1}f(0,0)$ states that

$$D_{2,1}f(0,0) = \lim_{k \to 0} \frac{D_1 f(0,k) - D_1 f(0,0)}{k}. \tag{1}$$

Now we have

$$D_1 f(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

and, if $(x,y) \neq (0,0)$, we find

$$D_1 f(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}.$$

Therefore, if $k \neq 0$ we have $D_1 f(0,k) = -k^5/k^4 = -k$ and hence

$$\frac{D_1 f(0,k) - D_1 f(0,0)}{k} = -1.$$

Using this in (1) we find that $D_{2,1} f(0,0) = -1$.

A similar argument shows that $D_{1,2} f(0,0) = 1$, and hence $D_{2,1} f(0,0) \neq D_{1,2} f(0,0)$.

Remark

- In the example just treated the two mixed partials $D_{1,2}f$ and $D_{2,1}f$ are not both continuous at the origin.
- It can be shown that the two mixed partials are equal at a point (a, b) if at least one of them is continuous in a neighborhood of the point.

Theorem (10.1)

Sufficient conditions for the equality of mixed partial derivatives

Assume f is a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, $D_{1,2} f$ and $D_{2,1} f$ exist on an open set $\mathbb{S}$. If (a, b) is a point in $\mathbb{S}$ at which both $D_{1,2} f$ and $D_{2,1} f$ are continuous, we have

$$D_{1,2} f(a, b) = D_{2,1} f(a, b). \tag{2}$$

Theorem (10.2)

Let f be a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, and $D_{2,1} f$ exist on an open set $\mathbb{S}$ containing (a, b). Assume further that $D_{2,1} f$ is continuous on $\mathbb{S}$. Then the derivative $D_{1,2} f(a, b)$ exists and we have

$$D_{1,2} f(a, b) = D_{2,1} f(a, b). \tag{3}$$

Thank you!