Real Analysis III

(MAT312β)

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References

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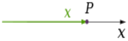
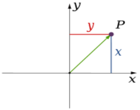
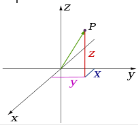
Introduction to *n*-dimensional space

What is dimension?

- In mathematics, the dimension of a space is informally defined as the minimum number of co-ordinates needed to specify any point within it.
- Thus a line has a dimension of one because only one co-ordinate is needed to specify a point on it.
- A plane has a dimension of two because two co-ordinates are needed to specify a point on it.
- The inside of a sphere is three-dimensional because three co-ordinates are needed to locate a point within this space.







Why do we need higher dimension?

- High-dimensional spaces occur in mathematics and the sciences for many reasons.
- For instance, if you are studying a chemical reaction involving 6 chemicals, you will probably want to store and manipulate their concentrations as a 6-tuple.
- The laws governing chemical reaction rates also demand we do calculus in this 6-dimensional space.

n-dimensional space

- We shall denote by $\mathbb{R}$ the field of real numbers.
- Then we shall use the Cartesian product $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \mathbb{R}$ of ordered n-tuples of real numbers (n factors).
- $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{x} = (x_1, x_2, \ldots, x_n)$.
- Here $\mathbf{x}$ is called a point or a vector, and $x_1, x_2, \ldots, x_n$ are called the coordinates of $\mathbf{x}$.
- The natural number n is called the dimension of the space.

n-dimensional space

Cont...

- $\mathbb{R}^1 \Rightarrow \mathbf{x} = (x_1)$
- $\mathbb{R}^2 \Rightarrow \mathbf{x} = (x_1, x_2)$
- $\mathbb{R}^3 \Rightarrow \mathbf{x} = (x_1, x_2, x_3)$
- $\mathbb{R}^4 \Rightarrow \mathbf{x} = (x_1, x_2, x_3, x_4)$
- $\mathbb{R}^m \Rightarrow \mathbf{x} = (x_1, x_2, ..., x_m)$
- $\mathbb{R}^n \Rightarrow \mathbf{x} = (x_1, x_2, ..., x_n)$

More on n-dimensional space

Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ be points in $\mathbb{R}^n$ and let a be a real number. Then we define

1. $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$.
2. $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ..., x_n - y_n)$.
3. $a\mathbf{x} = (ax_1, ax_2 ..., ax_n)$.

If $\mathbf{x} = (2, -3, 1)$ and $\mathbf{y} = (-4, 1, -2)$ are two points in $\mathbb{R}^3$, then find

(i) $\mathbf{x} + \mathbf{y}$.

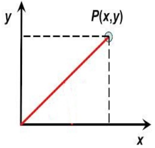
(ii) $\mathbf{x} - \mathbf{y}$.

(iii) $\mathbf{y} + \mathbf{x}$.

(iv) $2\mathbf{x} + 3\mathbf{y}$.

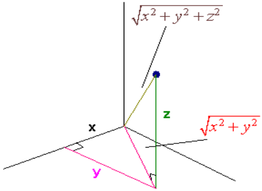
The length of a vector in two dimensional space

- We require some method to measure the magnitude of a vector.
- Based on Pythagorean Theorem, the vector from the origin to the point (4, 5) in two dimensional space has length of $\sqrt{4^2+5^2}=\sqrt{41}$.
- The vector from the origin to the point (x, y) has the length $\sqrt{x^2+y^2}$.
- The length of a vector with two elements is the square root of the sum of each element squared.



The length of a vector in three dimensional space

- The vector from the origin to the point (x, y, z) has the length $\sqrt{x^2 + y^2 + z^2}$.
- The length of a vector with three elements is the square root of the sum of each element squared.



The length of a vector in n-dimensional space

- In $\mathbb{R}^n$, the intuitive notion of length of the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is captured by the formula,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

- The magnitude of a vector is sometimes called the length of a vector, or **norm** of a vector.
- Basically, norm of a vector is a measure of distance, symbolized by $\|\mathbf{x}\|$.

The length of a vector in n-dimensional space
Example

Find the distances from the origin to the following vectors.

1. $\mathbf{x} = (2, 4, -1, 1) \in \mathbb{R}^4$
2. $\mathbf{y} = (1, 3, -2, 1, 4) \in \mathbb{R}^5$

The distance between two points in n-dimensional space

- In particular if we let $\|\mathbf{x}\|$ denote the distance from $\mathbf{x} = (x_1, x_2, ..., x_n)$ to the origin $\mathbf{0} = (0, 0, .., 0)$ in $\mathbb{R}^n$, then

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \\ \|\mathbf{x} - \mathbf{0}\| &= \sqrt{(x_1 - 0)^2 + (x_2 - 0)^2 + ... + (x_n - 0)^2}. \end{aligned}$$

- With this notation, the distance from $\mathbf{y} = (y_1, y_2, ..., y_n)$ to $\mathbf{x} = (x_1, x_2, ..., x_n)$ is

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

The distance between two points in n-dimensional space
Example

Let $\mathbf{x} = (1, 2, -3)$ and $\mathbf{y} = (3, -2, 1)$. Then find the distance from

(i) $\mathbf{x}$ to the origin.

(ii) $\mathbf{x}$ to $\mathbf{y}$.

Norm of a scalar times a vector

Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. If α is a scalar, how does the norm of $\alpha\mathbf{x}$ compare to the norm of $\mathbf{x}$?

Norm of a scalar times a vector

Proof

If $\mathbf{x} = (x_1, x_2, ..., x_n)$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$.

$$\begin{aligned} \|\alpha\mathbf{x}\| &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + ... + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2[(x_1)^2 + (x_2)^2 + ... + (x_n)^2]} \\ &= \sqrt{\alpha^2}.\sqrt{[(x_1)^2 + (x_2)^2 + ... + (x_n)^2]} \\ &= |\ \alpha|.\|\mathbf{x}\| \end{aligned}$$

Thus, multiplying a vector by a scalar α multiplies its norm by $|\ \alpha\ |$.

Unit vector

- Any vector whose length is 1 is called a unit vector.

- Let $\mathbf{x}$ be a given nonzero vector and consider the scalar multiple $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$.

- Applying the above result (with $\alpha = \frac{1}{\|\mathbf{x}\|}$), the norm of the vector $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is

$$\begin{aligned} \left\|\frac{1}{\|\mathbf{x}\|}\mathbf{x}\right\| &= \left|\frac{1}{\|\mathbf{x}\|}\right| \|\mathbf{x}\| \\ &= \frac{1}{\|\mathbf{x}\|}\|\mathbf{x}\| = 1 \end{aligned}$$

- Thus, for any nonzero vector $\mathbf{x}$, $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is a unit vector.

Unit vector
Example

Find the vector $\mathbf{v}$ in $\mathbb{R}^2$ whose length is 10 and which has the same direction as $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

First, find the unit vector in the same direction as $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, and then multiply this unit vector by 10. The unit vector in the direction of $\mathbf{u}$ is

$$\begin{aligned}
\hat{\mathbf{u}} &= \frac{\mathbf{u}}{\|\mathbf{u}\|} \\
&= \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} \\
&= \frac{3\mathbf{i} + 4\mathbf{j}}{5} \\
&= \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}
\end{aligned}$$

Therefore $\mathbf{v} = 10\hat{\mathbf{u}} = 10\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = 6\mathbf{i} + 8\mathbf{j}$.

Inner product

- An inner product is a generalization of the dot product.
- In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.
- The inner product is usually denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.
- In $\mathbb{R}^n$, where the inner product is given by the dot product,

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle (x_1, ..., x_n), (y_1, ..., y_n) \rangle \\ &= x_1 y_1 + x_2 y_2 + ... + x_n y_n \\ &= \sum_{k=1}^{n} x_k y_k \end{aligned}$$

Inner product
Example

What is the inner product of the vectors $\mathbf{x} = (-2, 1, 4, 1)$ and $\mathbf{y} = (1, 3, 2, 4)$ in $\mathbb{R}^4$?

Proposition

For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and scalar α. Then

1. $\mathbf{x}.\mathbf{y} = \mathbf{y}.\mathbf{x}$
2. $\mathbf{x}.(\mathbf{y} + \mathbf{z}) = \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}$
3. $(\alpha\mathbf{x}).\mathbf{y} = \alpha(\mathbf{x}.\mathbf{y})$
4. $\mathbf{0}.\mathbf{x} = 0$
5. $\mathbf{x}.\mathbf{x} \geq 0$
6. $\mathbf{x}.\mathbf{x} = \|\mathbf{x}\|^2$

Proposition
Poof of (1)

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n)$ and $\mathbf{z} = (z_1, z_2, ..., z_n)$.

$$\begin{aligned}
\mathbf{x}.\mathbf{y} &= (x_1, x_2, ..., x_n).(y_1, y_2, ..., y_n) \\
&= x_1 y_1 + x_2 y_2 + ... + x_n y_n \\
&= y_1 x_1 + y_2 x_2 + ... + y_n x_n \\
&= (y_1, y_2, ..., y_n).(x_1, x_2, ..., x_n) \\
&= \mathbf{y}.\mathbf{x}
\end{aligned}$$

Proposition
Poof of (2)

$$
\begin{aligned}
&= \mathbf{x}.(\mathbf{y}+\mathbf{z}) \\
&= (x_1, x_2, ..., x_n).[(y_1, y_2, ..., y_n) + (z_1, z_2, ..., z_n)] \\
&= (x_1, x_2, ..., x_n).(y_1 + z_1, y_2 + z_2, ..., y_n + z_n) \\
&= [x_1(y_1 + z_1) + x_2(y_2 + z_2) + ... + x_n(y_n + z_n)] \\
&= [(x_1y_1 + x_1z_1) + (x_2y_2 + x_2z_2) + ... + (x_ny_n + x_nz_n)] \\
&= (x_1y_1, x_2y_2, ..., x_ny_n) + (x_1z_1, x_2z_2, ..., x_nz_n) \\
&= [(x_1, x_2, ..., x_n).(y_1, y_2, ..., y_n)] + [(x_1, x_2, ..., x_n).(z_1, z_2, ..., z_n)] \\
&= \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}
\end{aligned}
$$

Proposition

Poof of (3)

$$
\begin{aligned}
(\alpha \mathbf{x})\mathbf{y} &= [\alpha(x_1, x_2, ..., x_n)].(y_1, y_2, ..., y_n) \\
&= (\alpha x_1, \alpha x_2, ..., \alpha x_n).(y_1, y_2, ..., y_n) \\
&= (\alpha x_1 y_1 + \alpha x_2 y_2 + ... + \alpha x_n y_n) \\
&= \alpha(x_1 y_1 + x_2 y_2 + ... + x_n y_n) \\
&= \alpha(\mathbf{x}.\mathbf{y})
\end{aligned}
$$

Proposition

Poof of (4)

$$
\begin{aligned}
\mathbf{0}.\mathbf{x} &= (0, 0, ..., 0).(x_1, x_2, ..., x_n) \\
&= 0x_1 + 0x_2 + ... + 0x_n \\
&= 0 + 0 + ... + 0 \\
&= 0
\end{aligned}
$$

Proposition
Poof of (5)

$$\begin{aligned} \mathbf{x}.\mathbf{x} &= (x_1, x_2, ..., x_n).(x_1, x_2, ..., x_n) \\ &= x_1^2 + x_2^2 + ... + x_n^2 \geq 0 \\ \text{Therefore } \mathbf{x}.\mathbf{x} &\geq 0 \end{aligned}$$

Proposition
Poof of (6)

$$\begin{aligned}\mathbf{x}.\mathbf{x} &= (x_1, x_2, ..., x_n).(x_1, x_2, ..., x_n)\\ \mathbf{x}.\mathbf{x} &= x_1^2 + x_2^2 + ... + x_n^2 \rightarrow \text{(A)}\\ \|\mathbf{x}\| &= \sqrt{x_1^2 + x_2^2 + ... + x_n^2}\\ \|\mathbf{x}\|^2 &= x_1^2 + x_2^2 + ... + x_n^2 \rightarrow \text{(B)}\end{aligned}$$

From (A) and (B)

$$\mathbf{x}.\mathbf{x} = \|\mathbf{x}\|^2$$

Direction cosine in $\mathbb{R}^3$

The direction cosines (or directional cosines) of a vector are the cosines of the angles between the vector and the three coordinate axes. If $\mathbf{u}$ is a vector

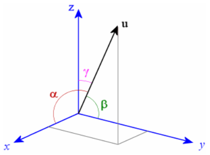
$$\mathbf{u} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$

then

$$\cos\alpha = \frac{x_1}{\|\mathbf{u}\|} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\cos\beta = \frac{x_2}{\|\mathbf{u}\|} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

$$\cos\gamma = \frac{x_3}{\|\mathbf{u}\|} = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$



Direction cosine in $\mathbb{R}^n$

- In general $\mathbf{x} \in \mathbb{R}^n$ can be considered as either a vector in $\mathbb{R}^n$ or as a point in $\mathbb{R}^n$ starting at the origin with length $\|\mathbf{x}\|$.
- If $\mathbf{x} \neq \mathbf{0}$ then we get

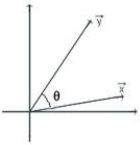
$$\mathbf{c} = \left(\frac{x_1}{\|\mathbf{x}\|}, \frac{x_2}{\|\mathbf{x}\|}, ..., \frac{x_n}{\|\mathbf{x}\|} \right)$$

  the direction of $\mathbf{x}$.
- The co-ordinates of $\mathbf{c}$, that is $\frac{x_k}{\|\mathbf{x}\|}$, $k = 1, 2, ..., n$ are called directional cosines.

The angle between two vectors

- Angle should take any two vectors **x** and **y** and produce a real number, $\theta \in [0, 2\pi)$.
- Angle should not depend on the lengths (norms) of **x** and **y**.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$\cos\theta = \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$



The angle between two vectors
Example

If $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -2, 2)$, find the angle between $\mathbf{x}$ and $\mathbf{y}$.

The angle between a vector and a co-ordinate axis

- Let $\mathbf{x} \in \mathbb{R}^n$, let $\alpha_k, k = 1, 2, 3, ..., n$ be the angle between $\mathbf{x}$ and the k^{th} axis.
- Then α_k is the angle between the standard basis vector $\mathbf{e}_k$ and $\mathbf{x}$.
- Thus we have

$$\cos(\alpha_k) = \frac{\mathbf{x}.\mathbf{e_k}}{\|\mathbf{x}\|\|\mathbf{e_k}\|} = \frac{x_k}{\|\mathbf{x}\|}.$$

The angle between a vector and a co-ordinate axis
Example

Find the angle between $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and the x axis.

Orthogonal vectors

- Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. Then if $\mathbf{x}.\mathbf{y} = 0$,

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= \frac{0}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= 0 \\
\cos\theta &= \cos\frac{\pi}{2} \\
\theta &= \frac{\pi}{2}
\end{aligned}$$

- The angle between $\mathbf{x}$ and $\mathbf{y}$ is $\frac{\pi}{2}$.

Orthogonal vectors
Cont...

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be orthogonal (or perpendicular) if $\mathbf{x}.\mathbf{y} = 0$.

$$\mathbf{x}.\mathbf{y} = 0 \implies \mathbf{x} \perp \mathbf{y}$$

Orthogonal vectors
Example

1. Show that $\mathbf{x} = (-1, -2)$ and $\mathbf{y} = (1, 2)$ are both orthogonal to $\mathbf{z} = (2, -1)$ in $\mathbb{R}^2$.
2. Show that $\mathbf{x} = (1, -1, 1, -1)$ and $\mathbf{y} = (1, 1, 1, 1)$ are perpendicular in $\mathbb{R}^4$.

Parallel vectors

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \neq 0$ is a scalar. Then we say that $\mathbf{x}$ and $\mathbf{y}$ are parallel if $\mathbf{x} = \alpha \mathbf{y}$.

$$\mathbf{x} = \alpha \mathbf{y} \Longrightarrow \mathbf{x} \parallel \mathbf{y}$$

Parallel vectors
Example

1. Suppose $\mathbf{x} = (2,1,3)$ and $\mathbf{y} = (4,2,6)$ in $\mathbb{R}^3$. We can write down $\mathbf{x} = \frac{1}{2}\mathbf{y}$. Therefore $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^3$.
2. Suppose $\mathbf{x} = (8,-2,6,-4)$ and $\mathbf{y} = (24,-6,18,-12)$ in $\mathbb{R}^4$. We can write down $\mathbf{y} = 3\mathbf{x}$. Therefore $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^4$.

Cauchy-Schwarz inequality

- The Cauchy-Schwarz inequality is a useful inequality encountered in many different situations.
- It is considered to be one of the most important inequalities in all of mathematics.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\| \tag{1}$$

is called the Cauchy-Schwarz inequality.

When $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, (1) holds with equality.

Let us assume that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are fixed vectors with $\mathbf{y} \neq \mathbf{0}$ (or $\mathbf{x} \neq \mathbf{0}$).

Take a real number $t \in \mathbb{R}$ and define the function

$$\begin{aligned} f(t) &= (\mathbf{x} + t\mathbf{y}).(\mathbf{x} + t\mathbf{y}) \\ &= (\mathbf{x} + t\mathbf{y})^2 \text{ (Therefore } f(t) \geq 0) \\ f(t) &= \mathbf{x}.\mathbf{x} + \mathbf{x}.t\mathbf{y} + t\mathbf{y}.\mathbf{x} + .t\mathbf{y}.t\mathbf{y} \\ f(t) &= \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y})t + t^2\|\mathbf{y}\|^2 \end{aligned}$$

Hence $f(t)$ is a quadratic function of t with at most one root.

The roots of $f(t)$ are given by

$$\frac{-2(\mathbf{x}.\mathbf{y}) \pm \sqrt{4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2}}{2\|\mathbf{y}\|^2}.$$

Since $f(t) \geq 0 \Longrightarrow 4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0.$

$$\begin{aligned} 4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 &\leq 0 \\ (\mathbf{x}.\mathbf{y})^2 &\leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \\ |\mathbf{x}.\mathbf{y}| &\leq \|\mathbf{x}\|\|\mathbf{y}\| \end{aligned}$$

$$\begin{aligned} |\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| &\iff f(t) = 0 \text{ for some value of } t \\ &\iff f(t) = 0 \\ &\iff (\mathbf{x} + t\mathbf{y})^2 = 0 \\ &\iff (\mathbf{x} + t\mathbf{y}) = 0 \\ &\iff \mathbf{x} = -t\mathbf{y} \end{aligned}$$

Moreover if $\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = \mathbf{0}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$.

Hence in either case the inequality (1) becomes an equality iff either $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$ or $\mathbf{y}$ is a scalar multiple of $\mathbf{x}$.

$$|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| \iff \mathbf{x} \parallel \mathbf{y}$$

$|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if $\mathbf{x}$ and $\mathbf{y}$ are parallel.

$$\begin{aligned}
|\mathbf{x}.\mathbf{y}| &= \|\mathbf{x}\|\|\mathbf{y}\| \\
\frac{|\mathbf{x}.\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} &= \frac{\|\mathbf{x}\|\|\mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\frac{|\mathbf{x}.\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} &= 1 \\
\frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} &= \pm 1 \longrightarrow (\mathrm{A}) \\
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \longrightarrow (\mathrm{B})
\end{aligned}$$

From (A) and (B)

$$\begin{aligned}\cos\theta &= \pm 1\\ \cos\theta &= 1 \qquad\qquad \cos\theta = -1\\ \cos\theta &= \cos 0 \qquad\quad \cos\theta = \cos\pi\\ \theta &= 0 \qquad\qquad\quad \theta = \pi\end{aligned}$$

Cauchy-Schwarz inequality

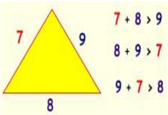
Remark 2⇒Example

Show that $\mathbf{x} = (1, -3)$ and $\mathbf{y} = (-2, 6)$ are parallel in $\mathbb{R}^2$.

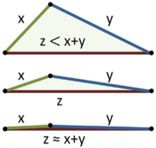
$$\begin{aligned}
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= \frac{(1,-3).(-2,6)}{\sqrt{1^2+(-3)^2}\sqrt{(-2)^2+6^2}} \\
\cos\theta &= \frac{-20}{2\times 10} \\
\cos\theta &= -1 \\
\cos\theta &= \cos\pi \\
\theta &= \pi \implies \mathbf{x} \parallel \mathbf{y}
\end{aligned}$$

Triangle inequality

- The triangle inequality states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side.
- The triangle inequality is a defining property of norms of vectors.
- That is, the norm of the sum of two vectors is at most as large as the sum of the norms of the two vectors.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$



$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$



Triangle inequality

The proof of the triangle inequality

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) \text{ (Since } \|\mathbf{x}\|^2 = \mathbf{x}.\mathbf{x}) \\
&= \mathbf{x}.\mathbf{x} + \mathbf{x}.\mathbf{y} + \mathbf{y}.\mathbf{x} + \mathbf{y}.\mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y}) + \|\mathbf{y}\|^2 \rightarrow \text{(A)} \\
|\mathbf{x}.\mathbf{y}| &\leq \|\mathbf{x}\|\|\mathbf{y}\| \text{ (CS inequality)} \\
\mathbf{x}.\mathbf{y} &\leq \|\mathbf{x}\|\|\mathbf{y}\| \rightarrow \text{(B)}
\end{aligned}
$$

From (A) and (B), we have

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + 2(\|\mathbf{x}\|\|\mathbf{y}\|) + \|\mathbf{y}\|^2 \\
\|\mathbf{x}+\mathbf{y}\|^2 &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\|\mathbf{x}+\mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}
$$

From (A)

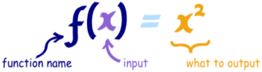
$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y}) + \|\mathbf{y}\|^2,$$

iff $\mathbf{x}.\mathbf{y} = 0$, that is iff $\mathbf{x}\perp\mathbf{y}$, then

$$\begin{aligned}
&\Rightarrow \quad \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(0) + \|\mathbf{y}\|^2 \\
&\Rightarrow \quad \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \\
&\Rightarrow \quad \text{Pythagorean theorem}
\end{aligned}$$

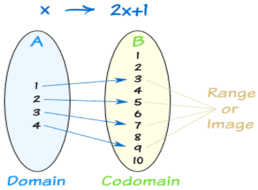
What is a function?

- In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.
- What can go into a function is called the **domain**.
- What may possibly come out of a function is called the **codomain**.
- What actually comes out of a function is called the **range**.



What is a function?
Example

- The set "A" is the Domain $\Rightarrow \{1, 2, 3, 4\}$.
- The set "B" is the Codomain $\Rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- The actual values produced by the function is the Range $\Rightarrow \{3, 5, 7, 9\}$.



Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

- We shall consider a function $\mathbf{f}$ with domain in n-space $\mathbb{R}^n$ and with range in m-space $\mathbb{R}^m$.
- It can be denoted as $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$.
- Both n and m are natural numbers and they can have different values.

- Then $f : \mathbb{R} \to \mathbb{R}$.
- Such a function is called as real-valued function of a real variable.
- In other words, it is a function that assigns a real number to each member of its domain.
- **Eg:** $f(x) = 2x + 1, f(x) = x^2 + 5, f(u) = 5u - 8$

- Then $\mathbf{f} : \mathbb{R} \to \mathbb{R}^m$.
- It is called as vector-valued function of a real variable.
- A common example of a vector valued function is one that depends on a single real number parameter t, often representing time, producing a vector $\mathbf{v}(t)$ as the result.
- **Eg:** $\mathbf{f}(t) = h(t)\mathbf{i} + g(t)\mathbf{j}$, where $h(t)$ and $g(t)$ are the coordinate functions of the parameter t.

- Then $f : \mathbb{R}^n \to \mathbb{R}$.
- The function is called as a real-valued function of a vector variable or, more briefly a **scalar field**.
- **Eg:** If $f : \mathbb{R}^3 \to \mathbb{R}$ the level surface of value c is the set of points $\{(x, y, z) : f(x, y, z) = c\}$.
- **Eg:** The temperature distribution throughout space, the pressure distribution in a fluid.

- Then $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$.
- The function is called as a vector-valued function of a vector variable or, more briefly a **vector field**.
- **Eg:** A function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ can be defined by

$$\mathbf{f}(x_1, x_2, x_3) = \left(\cos\sqrt{x_1^2 + x_2^2 + x_3^3 - 1}, \sin\sqrt{x_1^2 + x_2^2 + x_3^3 - 1}\right).$$

- **Eg:** Velocity field of a moving fluid, Magnetic fields, A gravitational field generated by any massive object.

- Scalars are denoted by light-faced characters.
- Vectors are denoted by bold-faced characters.
- If f is a scalar field defined at a point $\mathbf{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$, the notations $f(\mathbf{x})$ and $f(x_1, ..., x_n)$ are both used to denote the value of f at that particular point.
- If $\mathbf{f}$ is a vector field defined at a point $\mathbf{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$, the notations $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(x_1, ..., x_n)$ are both used to denote the value of $\mathbf{f}$ at that particular point.

Thank you !