Real Analysis III
(MAT312β)

Department of Mathematics
University of Ruhuna

A.W.L. Pubudu Thilan

Derivatives of Functions Defined Implicitly

Explicit function

A function in which the dependent variable can be written explicitly in terms of the independent variable.

Eg:

(a) $y = x^3 + 9$

(b) $y = \sqrt{4 - x^2}$

(c) $y = \log_5 x$

Implicit function

- A function or relation in which the dependent variable is not isolated on one side of the equation.
- Some implicit functions can be written explicitly.
- Unfortunately, not every equation involving x and y can be solved explicitly for y.

Eg:

(a) $x^2 + y^2 = 4$
(b) $y - x^2 = 13$
(c) $y^5 + x^4 y^7 - 2x^4 y + x^5 = 0$

Implicit differentiation of functions of one variable

- We have seen how to differentiate functions of the form $y = f(x)$.
- We also want to be able to differentiate functions that either can't be written explicitly in terms of x or the resulting function is too complicated to deal with.
- To do this we use implicit differentiation.
- Implicit differentiation is nothing more than a special case of the well-known chain rule for derivatives.

(a) Find $\frac{dy}{dx}$ for $x^2 + y^2 = 4$

(b) Find $\frac{dy}{dx}$ for $x^2y + y^3x = x^3y^3$

Implicit representation of the surface in 3-space

- The equation $x^2 + y^2 + z^2 - 1 = 0$ represents the surface of a unit sphere center at the origin.
- It is an equation of the form

$$F(x, y, z) = 0.$$

- An equation like this is said to provide an implicit representation of the surface.

- Sometimes it is possible to solve the equation $F(x, y, z) = 0$ for one of the variables in terms of the other two, say for z in terms of x and y.

- This leads to one or more equations of the form

$$z = f(x, y).$$

- For the sphere we have two solution,

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2},$$

one representing the upper hemisphere, the other the lower hemisphere.

Implicit differentiation of functions of two variables

- In the general case it may not be an easy matter to obtain an explicit formula for z in terms of x and y.
- For example, there is no easy method for solving for z in the equation $y^2 + xz + z^2 - e^z - 4 = 0$.
- Nevertheless, a judicious use of the chain rule makes it possible to deduce various properties of the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ without an explicit knowledge of $f(x, y)$.
- The procedure is described in this Chapter.

- We assume that there is a function $f(x, y)$ such that

$$F[x, y, f(x, y)] = 0 \tag{1}$$

for all (x, y) in some open set S, although we may not have explicit formula for calculating $f(x, y)$.

- We describe this by saying that the equation $F(x, y, z) = 0$ defines z implicitly as a function of x and y, and we write

$$z = f(x, y).$$

- Now we introduce an auxiliary function g defined on S as follows:

$$g(x, y) = F[x, y, f(x, y)].$$

- Equation (1) states that $g(x, y) = 0$ on S; hence the partial derivatives $\partial g/\partial x$ and $\partial g/\partial y$ are also 0 on S.

Implicit differentiation of functions of two variables
Cont...

- But we can also compute these partial derivatives by the chain rule.
- To do this we write

$$g(x, y) = F[u_1(x, y), u_2(x, y), u_3(x, y)],$$

where $u_1(x, y) = x$, $u_2(x, y) = y$, and $u_3(x, y) = f(x, y)$.

The chain rule gives us the formulas

$$\frac{\partial g}{\partial x} = D_1 F \frac{\partial u_1}{\partial x} + D_2 F \frac{\partial u_2}{\partial x} + D_3 F \frac{\partial u_3}{\partial x}$$

and

$$\frac{\partial g}{\partial y} = D_1 F \frac{\partial u_1}{\partial y} + D_2 F \frac{\partial u_2}{\partial y} + D_3 F \frac{\partial u_3}{\partial y}$$

where each partial derivative $D_k F$ is to be evaluated at $(x, y, f(x, y))$.

Implicit differentiation of functions of two variables
Cont...

Since we have

$$\frac{\partial u_1}{\partial x} = 1, \ \frac{\partial u_2}{\partial x} = 0, \ \frac{\partial u_3}{\partial x} = \frac{\partial f}{\partial x} \text{ and } \frac{\partial g}{\partial x} = 0,$$

the first of the foregoing equations becomes

$$D_1 F + D_3 F \frac{\partial f}{\partial x} = 0.$$

Solving this for $\partial f / \partial x$ we obtain

$$\frac{\partial f}{\partial x} = -\frac{D_1 F[x, y, f(x, y)]}{D_3 F[x, y, f(x, y)]} \tag{2}$$

at those points at which $D_3 F[x, y, f(x, y)] \neq 0$.

By a similar arguments we obtain a corresponding formula for $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -\frac{D_2F[x,y,f(x,y)]}{D_3F[x,y,f(x,y)]} \tag{3}$$

at those points at which $D_3F[x,y,f(x,y)] \neq 0$.
These formulas are usually written more briefly as follows:

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\frac{\partial F/\partial x}{\partial F/\partial z}\\ \frac{\partial f}{\partial y} &= -\frac{\partial F/\partial y}{\partial F/\partial z}\end{aligned}$$

Example 1

Assume that the equation $y^2 + xz + z^2 - e^z - c = 0$ defines z as a function of x and y, say $z = f(x, y)$. Find a value of the constant c such that $f(0, e) = 2$, and compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(x, y) = (0, e)$.

Example 1
Solution

When $x = 0$, $y = e$, and $z = 2$, the equation becomes $e^2 + 4 - e^2 - c = 0$, and this is satisfied by $c = 4$. Let $F(x, y, z) = y^2 + xz + z^2 - e^z - 4$. From (2) and (3) we have

$$\frac{\partial f}{\partial x} = -\frac{z}{x + 2z - e^z}, \quad \frac{\partial f}{\partial y} = -\frac{2y}{x + 2z - e^z}.$$

When $x = 0, y = e$, and $z = 2$ we find $\partial f/\partial x = 2/(e^2 - 4)$ and $\partial f/\partial y = 2e/(e^2 - 4)$.

Note that we were able to compute the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ using only the value of $f(x, y)$ at the single point $(0, e)$.

Example 2
Past paper 2013

(i) Consider a surface in 3-space is described by the equation $F(x, y, z) = 0$, where $z = f(x, y)$ is implicitly defined as a function of x and y. Write down the expressions for $\partial z/\partial x$ and $\partial z/\partial y$.

(ii) Suppose that

$$\sin xy + \sin yz + \sin zx = 1,$$

defines the variable z as a function of x and y. Then find $\partial z/\partial x$ and $\partial z/\partial y$.

Example 2
Solution

(i)

$$\frac{\partial f}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$
$$\frac{\partial f}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Example 2
Solution

(ii)

$$
\begin{aligned}
F(x,y,z) &= \sin xy + \sin yz + \sin zx - 1 \\
\frac{\partial F}{\partial x} &= y\cos xy + z\cos zx \\
\frac{\partial F}{\partial z} &= y\cos yz + x\cos zx \\
\frac{\partial F}{\partial y} &= x\cos xy + z\cos yz \\
\frac{\partial f}{\partial x} &= -\frac{y\cos xy + z\cos zx}{y\cos yz + x\cos zx} \\
\frac{\partial f}{\partial y} &= -\frac{x\cos xy + z\cos yz}{y\cos yz + x\cos zx}
\end{aligned}
$$

Theorem (5.1)

Implicit differentiation of functions of more than two variables

Let F be a scalar field differentiable on an open set $\mathbb{T}$ in $\mathbb{R}^n$. Assume that the equation

$$F(x_1, ..., x_n) = 0$$

defines x_n implicitly as a differentiable function of $x_1, ..., x_{n-1}$, say

$$x_n = f(x_1, ..., x_{n-1}),$$

for all points $(x_1, ..., x_{n-1})$ in some open set $\mathbf{S}$ in $\mathbb{R}^{n-1}$. Then for each $k = 1, 2, ..., n-1$ the partial derivative $D_k f$ is given by the formula

$$D_k f = -\frac{D_k F}{D_n F} \tag{4}$$

at those points at which $D_n F \neq 0$. The partial derivatives $D_k F$ and $D_n F$ which appear in (4) are to be evaluated at the point $(x_1, x_2, ..., x_{n-1}, f(x_1, ..., x_{n-1}))$.

Two surfaces having implicit representations

- Suppose we have two surfaces with the following implicit representations:

$$F(x, y, z) = 0, \quad G(x, y, z) = 0. \tag{5}$$

- If these surfaces intersects along a curve C, it may be possible to obtain a parametric representation of C by solving the two equations in (5) simultaneously for two of the variables in term of the third, say for x and y in terms of z.

Two surfaces having implicit representations
Cont...

- Let us suppose that it is possible to solve for x and y and that solutions are given by the equations

$$x = X(z), \quad y = Y(z)$$

for all z in some open interval (a, b).

- Then when x and y are replaced by $X(z)$ and $Y(z)$ respectively, the two equations in (5) are identically satisfied.

- That is, we can write $F[X(z), Y(z), z] = 0$ and $G[X(z), Y(z), z] = 0$ for all z in (a, b).

Two surfaces having implicit representations
Cont...

- Again, by using the chain rule, we can compute the derivatives $X'(z)$ and $Y'(z)$ without an explicit knowledge of $X(z)$ and $Y(z)$.

- To do this we introduce new functions f and g by means of the equations

$$f(z) = F[X(z), Y(z), z] \text{ and } g(z) = G[X(z), Y(z), z].$$

- Then $f(z) = g(z) = 0$ for every z in (a, b) and hence the derivatives $f'(z)$ and $g'(z)$ are also zero on (a, b).

- By the chain rule these derivatives are given by the formula

$$f'(z) = \frac{\partial F}{\partial x} X'(z) + \frac{\partial F}{\partial y} Y'(z) + \frac{\partial F}{\partial z},$$
$$g'(z) = \frac{\partial G}{\partial x} X'(z) + \frac{\partial G}{\partial y} Y'(z) + \frac{\partial G}{\partial z}.$$

- Since $f'(z)$ and $g'(z)$ are both zero we can determine $X'(z)$ and $Y'(z)$ by solving the following pair of simultaneous linear equations:

$$\frac{\partial F}{\partial x}X'(z) + \frac{\partial F}{\partial y}Y'(z) = -\frac{\partial F}{\partial z},$$
$$\frac{\partial G}{\partial x}X'(z) + \frac{\partial G}{\partial y}Y'(z) = -\frac{\partial G}{\partial z}.$$

- At those points at which the determinant of the system is not zero, these equations have a unique solution which can be expressed as follows, using Cramer's rule:

Two surfaces having implicit representations
Cont...

$$X'(z) = -\frac{\begin{vmatrix} \dfrac{\partial F}{\partial z} & \dfrac{\partial F}{\partial y} \\[2ex] \dfrac{\partial G}{\partial z} & \dfrac{\partial G}{\partial y} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\[2ex] \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix}}, \quad Y'(z) = -\frac{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial z} \\[2ex] \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial z} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\[2ex] \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix}} \tag{6}$$

Two surfaces having implicit representations
Cont...

- The determinants which appear in (6) are determinants of Jacobian matrices and are called Jacobian determinants.
- A special notation is often used to denote Jacobian determinants.
- We write

$$\frac{\partial(f_1, ..., f_n)}{\partial(x_1, ..., x_n)} = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdot \quad \cdot \quad \cdot & \frac{\partial f_1}{\partial x_n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdot \quad \cdot \quad \cdot & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

In this notation, the formulas in (6) can be expressed more briefly in the form

$$X'(z) = \frac{\partial(F,G)/\partial(y,z)}{\partial(F,G)/\partial(x,y)}, Y'(z) = \frac{\partial(F,G)/\partial(z,x)}{\partial(F,G)/\partial(x,y)}. \tag{7}$$

(The minus sign has been incorporated into the numerators by interchanging the columns)

Two surfaces having implicit representations

Cont...

- The method can be extended to treat more general situations in which m equations in n variables are given, where $n > m$ and we solve for m of the variables in terms of the remaining $n - m$ variables.
- The partial derivatives of the new functions so defined can be expressed as quotients of the Jacobian determinants, generalizing (7).

Example 1

Assume that the equation $g(x, y) = 0$ determines y as a differentiable function of x , say $y = Y(x)$ for all x in some open interval (a, b). Express the derivative $Y'(x)$ in terms of the partial derivatives of g.

Example 1
Solution

Let $G(x) = g[x, Y(x)]$ for x in (a, b). Then the equation $g(x, y) = 0$ implies $G(x) = 0$ in (a, b). By the chain rule we have

$$G'(x) = \frac{\partial g}{\partial x}.1 + \frac{\partial g}{\partial y} Y'(x),$$

from which we obtain

$$Y'(x) = -\frac{\partial g/\partial x}{\partial g/\partial y} \tag{8}$$

at those points x in (a, b) at which $\partial g/\partial y \neq 0$. The partial derivatives $\partial g/\partial x$ and $\partial g/\partial y$ are given by the formulas $\partial g/\partial x = D_1 g[x, Y(x)]$ and $\partial g/\partial y = D_2 g[x, Y(x)]$.

Example 2

When y is eliminated from the two equations $z = f(x, y)$ and $g(x, y) = 0$, the results can be expressed in the form $z = h(x)$. Express the derivative $h'(x)$ in terms of the partial derivatives of f and g.

Let us assume that the equation $g(x, y) = 0$ may be solved for y in terms of x and that a solution is given by $y = Y(x)$ for all x in some open interval (a, b), Then the function h is given by the formula

$$h(x) = f[x, Y(x)] \quad \text{if } x \in (a, b).$$

Applying the chain rule we have

$$h'(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} Y'(x).$$

Using Equation (8) of Example 1 we obtain the formula

$$h'(x) = \frac{\dfrac{\partial g}{\partial y}\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}}{\dfrac{\partial g}{\partial y}}.$$

The partial derivatives on the right are to be evaluated at the point $(x, Y(x))$. Note that the numerator can also be expressed as a Jacobian determinant, giving us

$$h'(x) = \frac{\partial(f, g)/\partial(x, y)}{\partial g/\partial y}.$$

Example 3

Let u be defined as a function of x and y by means of the equation

$$u = F(x + u, yu). \tag{9}$$

Find $\partial u/\partial x$ and $\partial u/\partial y$ in terms of the partial derivatives of F.

Example 3

Solution

Suppose that $u = g(x, y)$ for all (x, y) in some open set S. Substituting $g(x, y)$ for u in the original equation we must have

$$g(x, y) = F[u_1(x, y), u_2(x, y)], \tag{10}$$

where $u_1(x, y) = x + g(x, y)$ and $u_2(x, y) = yg(x, y)$. Now we hold y fixed and differentiate both sides of (10) with respect to x, using the chain rule on the right, to obtain

$$\frac{\partial g}{\partial x} = D_1 F \frac{\partial u_1}{\partial x} + D_2 F \frac{\partial u_2}{\partial x}. \tag{11}$$

But $\partial u_1/\partial x = 1 + \partial g/\partial x$, and $\partial u_2/\partial x = y\partial g/\partial x$. Hence (11) becomes

$$\frac{\partial g}{\partial x} = D_1F.\left(1 + \frac{\partial g}{\partial x}\right) + D_2F.\left(y\frac{\partial g}{\partial x}\right).$$

Solving this equation for $\partial g/\partial x$ (and writing $\partial u/\partial x$ for $\partial g/\partial x$) we obtain

$$\frac{\partial u}{\partial x} = \frac{-D_1F}{D_1F + yD_2F - 1}.$$

In a similar way we find

$$\frac{\partial g}{\partial y} = D_1 F \frac{\partial u_1}{\partial y} + D_2 F \frac{\partial u_2}{\partial y} = D_1 F \frac{\partial g}{\partial y} + D_2 F \left(y \frac{\partial g}{\partial y} + g(x, y) \right).$$

This leads to the equation

$$\frac{\partial u}{\partial y} = \frac{-g(x, y) D_2 F}{D_1 F + y D_2 F - 1}.$$

The partial derivatives $D_1 F$ and $D_2 F$ are to be evaluated at the point $(x + g(x, y), yg(x, y))$.

Example 4

When u is eliminated from the two equations $x = u + v$ and $y = uv^2$, we get an equation of the form $F(x, y, v) = 0$ which defines v implicitly as a function of x and y, say $v = h(x, y)$. Prove that

$$\frac{\partial h}{\partial x} = \frac{h(x, y)}{3h(x, y) - 2x}$$

and find a similar formula for $\partial h / \partial y$.

Eliminating u from the given two equations, we obtain the relation

$$xv^2 - v^3 - y = 0.$$

Let F be the function defined by the equation

$$F(x, y, v) = xv^2 - v^3 - y.$$

By using the two formulas we can write

$$\frac{\partial h}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial v} \quad \text{and} \quad \frac{\partial h}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial v}. \tag{12}$$

Example 4
Solution⇒Cont...

But $\partial F/\partial x = v^2$, $\partial F/\partial v = 2xv - 3v^2$, and $\partial F/\partial y = -1$. Hence the equation becomes

$$\begin{aligned}\frac{\partial h}{\partial x} &= -\frac{v^2}{2xv - 3v^2} \\ &= -\frac{v}{2x - 3v} \\ &= \frac{h(x,y)}{3h(x,y) - 2x}.\end{aligned}$$

$$\begin{aligned}\frac{\partial h}{\partial y} &= -\frac{-1}{2xv - 3v^2} \\ &= \frac{1}{2xh(x,y) - 3h^2(x,y)}.\end{aligned}$$

Exercise

Suppose that

$$x^2y^2z^3 + zx\sin y = 5$$

defines z as a function of x and y. Then find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Thank you!