Real Analysis III
(MAT312β)

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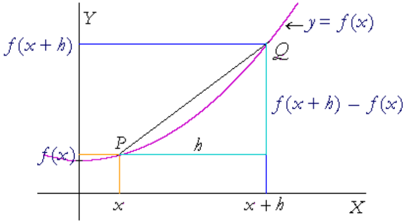
Derivatives of Functions of Several Variables I

The Derivative of a Scalar Field with Respect to a Vector

Definition
The derivative of a single variable function

The derivative of the function $f(x)$ at the point x is given and denoted by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



Why do we need a vector to get derivative of a scalar field?

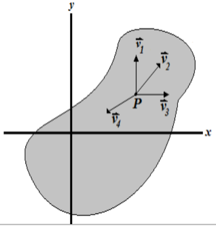
- Suppose $y = f(x)$. Then the derivative $f'(x)$ is the rate at which y changes when we let x vary.
- Since f is a function on the real line, so the variable can only increase or decrease along that single line.
- In one dimension, there is only one "direction" in which x can change.

Why do we need a vector to get derivative of a scalar field?
Cont...

- Given a function of two or more variables like $z = f(x, y)$, there are infinitely many different directions from any point in which the function can change.
- We know that we can represent directions by using vectors.
- Derivative of a scalar field is the rate of change of the scalar field in a particular direction given by a vector.

Why do we need a vector to get derivative of a scalar field?
Cont...

- Let P is a point in the domain of $f(x, y)$ and vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, and $\mathbf{v_4}$ represent possible directions in which we might want to know the rate of change of $f(x, y)$.
- Suppose we may want to know the rate at which $f(x, y)$ is changing along or in the direction of the vector, $\mathbf{v_3}$, which would be the direction along the x-axis.



The derivative of a scalar field with respect to a vector
Motivative example

- Suppose a person is at point $\mathbf{a}$ in a heated room with an open window.
- Let $f(\mathbf{a})$ is the temperature at a point $\mathbf{a}$.
- If the person moves toward the window temperature will decrease, but if the person moves toward heater it will increase.
- In general, the manner in which a field changes will depend on the direction in which we move away from $\mathbf{a}$.

- Let $f : \mathbf{S} \to \mathbb{R}$ be a scalar field where $\mathbf{S} \subseteq \mathbb{R}^n$ and let $\mathbf{a}$ be an interior point of $\mathbf{S}$.
- We are going to study about how the field changes as we move from $\mathbf{a}$ to a nearby point.

- Suppose moving direction is given by the vector $\mathbf{y}$.
- That is suppose we move from $\mathbf{a}$ toward another point $\mathbf{a}+\mathbf{y}$ along the line segment joining $\mathbf{a}$ and $\mathbf{a}+\mathbf{y}$.
- Each point on this segment has the form $\mathbf{a}+h\mathbf{y}$, where h is a real number.
- The distance from $\mathbf{a}$ to $\mathbf{a}+h\mathbf{y}$ is $\|h\mathbf{y}\| = |h|\|\mathbf{y}\|$.

- Since $\mathbf{a}$ is an interior point of $\mathbf{S}$, there is an n-ball $\mathbf{B}(\mathbf{a};\mathbf{r})$ lying entirely in $\mathbf{S}$.
- If h is chosen so that $|h|\|\mathbf{y}\| < r$, the segment from $\mathbf{a}$ to $\mathbf{a} + h\mathbf{y}$ will lie in $\mathbf{S}$.
- We keep $h \neq 0$ but small enough to guarantee that $\mathbf{a} + h\mathbf{y} \in \mathbf{S}$.
- So, then from the difference quotient we have,

$$\frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h}.$$

- If we consider the above quotient, the numerator tells us how much the function changes when we move from $\mathbf{a}$ to $\mathbf{a} + h\mathbf{y}$.
- The quoteint itself is called the **average rate of change** of f over the line segmengnt joining $\mathbf{a}$ to $\mathbf{a} + h\mathbf{y}$.
- We are interested in the behavior of this quotient as $h \to 0$.
- This leads us to the following definition.

Definition

The derivative of a scalar field with respect to a vector

Given a scalar field $f : \mathbf{S} \to \mathbb{R}$, where $\mathbf{S} \subseteq \mathbb{R}^n$. Let $\mathbf{a}$ be an interior point of $\mathbf{S}$ and let $\mathbf{y}$ be an arbitrary point in $\mathbb{R}^n$. The derivative of f at $\mathbf{a}$ with respect to $\mathbf{y}$ is denoted by the symbol $f'(\mathbf{a}; \mathbf{y})$ and is defined by the equation

$$f'(\mathbf{a}; \mathbf{y}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h}, \tag{1}$$

when the limit on the right exists.

Example 1

If $\mathbf{y} = \mathbf{0}$, the difference quotient (1) is 0 for every $h \neq 0$, so $f'(\mathbf{a}; \mathbf{0})$ always exists and equals 0.

$$
\begin{aligned}
f'(\mathbf{a}; \mathbf{y}) &= \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a}; \mathbf{0}) &= \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{0}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a}; \mathbf{0}) &= \lim_{h \to 0} \frac{f(\mathbf{a}) - f(\mathbf{a})}{h} \\
&= 0.
\end{aligned}
$$

Example 2
Derivative of a linear transformation

If $f : \mathbf{S} \to \mathbb{R}$ is a linear transformation, then $f(\mathbf{a} + h\mathbf{y}) = f(\mathbf{a}) + hf(\mathbf{y})$. From the definition we have,

$$\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{f(\mathbf{a}) + hf(\mathbf{y}) - f(\mathbf{a})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0} \frac{hf(\mathbf{y})}{h} \\
f'(\mathbf{a};\mathbf{y}) &= f(\mathbf{y}).
\end{aligned}$$

Therefore, the derivative of linear transformation with respect to $\mathbf{y}$ is equal to the value of the function at $\mathbf{y}$.

Example 3

A scalar field f is defined on $\mathbb{R}^n$ by the equation $f(\mathbf{x}) = \mathbf{a}.\mathbf{x}$, where $\mathbf{a}$ is a constant vector. Compute $f'(\mathbf{x}; \mathbf{y})$ for arbitary $\mathbf{x}$ and $\mathbf{y}$.

Example 3
Solution

According to the definition, we have

$$\begin{aligned} f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} \\ f'(\mathbf{x};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{y})-f(\mathbf{x})}{h} \\ &= \lim_{h\to 0}\frac{\mathbf{a}.(\mathbf{x}+h\mathbf{y})-\mathbf{a}.\mathbf{x}}{h} \\ &= \lim_{h\to 0}\frac{h(\mathbf{a}.\mathbf{y})}{h} \\ &= \mathbf{a}.\mathbf{y}. \end{aligned}$$

Example 4

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a given linear transformation. Compute the derivative $f'(\mathbf{x}; \mathbf{y})$ for the scalar field defined on $\mathbb{R}^n$ by the equation $f(\mathbf{x}) = \mathbf{x}.T(\mathbf{x})$.

Example 4
Solution

According to the definition, we have

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h}\\
f'(\mathbf{x};\mathbf{y}) &= \lim_{h\to 0}\frac{f(\mathbf{x}+h\mathbf{y})-f(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{(\mathbf{x}+h\mathbf{y}).T(\mathbf{x}+h\mathbf{y})-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{(\mathbf{x}+h\mathbf{y}).(T(\mathbf{x})+hT(\mathbf{y}))-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{\mathbf{x}.T(\mathbf{x})+h\mathbf{x}.T(\mathbf{y})+h\mathbf{y}.T(\mathbf{x})+h^2\mathbf{y}.T(\mathbf{y})-\mathbf{x}.T(\mathbf{x})}{h}\\
&= \lim_{h\to 0}\frac{h\mathbf{x}.T(\mathbf{y})+h\mathbf{y}.T(\mathbf{x})+h^2\mathbf{y}.T(\mathbf{y})}{h}\\
&= \mathbf{x}.T(\mathbf{y})+\mathbf{y}.T(\mathbf{x}).
\end{aligned}
$$

Pre-requisite for theorem 3.1

To study how f behaves on the line passing through $\mathbf{a}$ and $\mathbf{a} + \mathbf{y}$ for $\mathbf{y} \neq \mathbf{0}$ we introduce the function

$$g(t) \quad = \quad f(\mathbf{a} + t\mathbf{y}).$$

The next theorem relates the derivatives $g'(t)$ and $f'(\mathbf{a} + t\mathbf{y}; \mathbf{y})$.

Theorem 3.1

Let $g(t) = f(\mathbf{a} + t\mathbf{y})$. If one of the derivatives $g'(t)$ or $f'(\mathbf{a} + t\mathbf{y}; \mathbf{y})$ exists then the other also exists and the two are equal,

$$g'(t) \quad = \quad f'(\mathbf{a} + t\mathbf{y}; \mathbf{y}).$$

In particular, when $t = 0$ we have $g'(0) = f'(\mathbf{a}; \mathbf{y})$.

Theorem 3.1

Proof

Forming the difference quotient for g, we have,

$$\begin{aligned}
\frac{g(t+h)-g(t)}{h} &= \frac{f(\mathbf{a}+(t+h)\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
\frac{g(t+h)-g(t)}{h} &= \frac{f(\mathbf{a}+t\mathbf{y}+h\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} &= \lim_{h\to 0}\frac{f(\mathbf{a}+t\mathbf{y}+h\mathbf{y})-f(\mathbf{a}+t\mathbf{y})}{h}\\
g'(t) &= f'(\mathbf{a}+t\mathbf{y};\mathbf{y}).
\end{aligned}$$

Example

Compute $f'(\mathbf{a}; \mathbf{y})$ if $f(\mathbf{x}) = \|\mathbf{x}\|^2$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Example
Solution

We let

$$
\begin{aligned}
g(t) &= f(\mathbf{a} + t\mathbf{y}) \\
&= \|\mathbf{a} + t\mathbf{y}\|^2 \text{ since } f(\mathbf{x}) = \|\mathbf{x}\|^2 \\
&= (\mathbf{a} + t\mathbf{y}).(\mathbf{a} + t\mathbf{y}) \text{ since } \|\mathbf{x}\|^2 = \mathbf{x}.\mathbf{x} \\
&= \mathbf{a}.\mathbf{a} + t\mathbf{a}.\mathbf{y} + t\mathbf{y}.\mathbf{a} + t^2\mathbf{y}.\mathbf{y} \\
g(t) &= \mathbf{a}.\mathbf{a} + 2t\mathbf{a}.\mathbf{y} + t^2\mathbf{y}.\mathbf{y} \\
g'(t) &= 0 + 2\mathbf{a}.\mathbf{y} + 2t\mathbf{y}.\mathbf{y}
\end{aligned}
$$

We need to find $f'(\mathbf{a};\mathbf{y})$. If we subsitute

$$
\begin{aligned}
f'(\mathbf{a} + 0\mathbf{y};\mathbf{y}) &= g'(0) = 2\mathbf{a}.\mathbf{y} \\
f'(\mathbf{a};\mathbf{y}) &= 2\mathbf{a}.\mathbf{y}.
\end{aligned}
$$

Directional Derivatives and Partial Derivatives

Directional Derivatives

- As mentioned above, given a function of two or more variables like $z = f(x, y)$, there are infinitely many different directions from any point in which the function can change.
- We know that we can represent directions as vectors, particularly unit vectors when its only the direction and not the magnitude that concerns us.
- Directional derivatives are literally just derivatives or rates of change of a function in a particular direction given by a unit vector.

If $\mathbf{u}$ is a unit vector, then

$$f'(\mathbf{a};\mathbf{u}) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{u}) - f(\mathbf{a})}{h},$$

the derivative $f'(\mathbf{a};\mathbf{u})$ is called the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{u}$.

Directional derivative of $f(x, y)$ at (a, b) in the direction of $\mathbf{u}$

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, we define the directional derivative $f_{\mathbf{u}}$ at the point (a, b) by

$$\begin{aligned} f_{\mathbf{u}}(a, b) &= \text{Rate of change of } f(x, y) \text{ in the direction of } \mathbf{u} \\ &\quad \text{at the point } (a, b) \\ &= \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} \end{aligned}$$

provided that the limit exists.

Example 1

Compute the directional derivative of $f(x, y) = x + y^2$ at the point (4, 0) in the direction $\mathbf{u} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$.

Example 1
Solution

The norm of $\mathbf{u}$, that is $\|\mathbf{u}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$ Thus $\mathbf{u}$ is a unit vector.

$$\begin{aligned}
f_{\mathbf{u}}(4,0) &= \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} \\
&= \lim_{h\to 0} \frac{f(4+h\frac{1}{2}, 0+h\frac{\sqrt{3}}{2}) - f(4,0)}{h} \\
&= \lim_{h\to 0} \frac{(4+h\frac{1}{2}) + \left(h\frac{\sqrt{3}}{2}\right)^2 - 4}{h} \\
&= \lim_{h\to 0} \frac{4+\frac{h}{2}+\frac{3h^2}{4} - 4}{h} = \lim_{h\to 0}\left(\frac{1}{2}+\frac{3}{4}h\right) = \frac{1}{2}
\end{aligned}$$

Partial derivatives

- If $\mathbf{u}$ is a unit vector, the derivative $f'(\mathbf{a};\mathbf{u})$ is called the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{u}$.

- In particular, if $\mathbf{u} = \mathbf{e}_k$ (the k^{th} unit coordinate vector) the directional derivative $f'(\mathbf{a};\mathbf{e}_k)$ is called partial derivative with respect to $\mathbf{e}_k$ and is also denoted by the symbool $D_k f(\mathbf{a})$.

- Thus

$$
\begin{aligned}
f'(\mathbf{a};\mathbf{u}) &= \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{u}) - f(\mathbf{a})}{h}, \\
f'(\mathbf{a};\mathbf{e}_k) &= \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{e}_k) - f(\mathbf{a})}{h} = D_k f(\mathbf{a}).
\end{aligned}
$$

The following notations are also used for the partial derivative $D_k f(\mathbf{a})$:

(i) $D_k f(a_1, ..., a_n)$,

(ii) $\dfrac{\partial f}{\partial x_k}(a_1, ..., a_n)$,

(iii) $f'_{x_k}(a_1, ..., a_n)$.

Sometimes the derivative f'_{x_k} is written without the prime as f_{x_k} or even more simply as f_k.

- In $\mathbb{R}^2$ the unit coordinate vectors are denoted by $\mathbf{i}$ and $\mathbf{j}$.
- If $\mathbf{a} = (a, b)$ the partial derivatives $f'(\mathbf{a}; \mathbf{i})$ and $f'(\mathbf{a}; \mathbf{j})$ are also written as

$$\frac{\partial f}{\partial x}(a, b) \text{ and } \frac{\partial f}{\partial y}(a, b),$$

respectively.

Consider the function $f(x, y) = 9 - x^2 - y^2$. Let's investigate $f_x(1, 2)$.

We fix $y = 2$ and construct the single variable function $g(x) = f(x, 2) = 9 - x^2 - 2^2 = 5 - x^2$. This parabola lies on the paraboloid $f(x, y) = 9 - x^2 - y^2$ and in the vertical plane $y = 2$.

Now, $g'(x) = -2x$ and so $f_x(1, 2) = g'(1) = -2(1) = -2$. This should be the slope of the tangent line to this curve $g(x) = 5 - x^2$ lying in the vertical plane $y = 2$.

Partial derivatives in $\mathbb{R}^2$

Notations ⇒ Example ⇒ Cont...

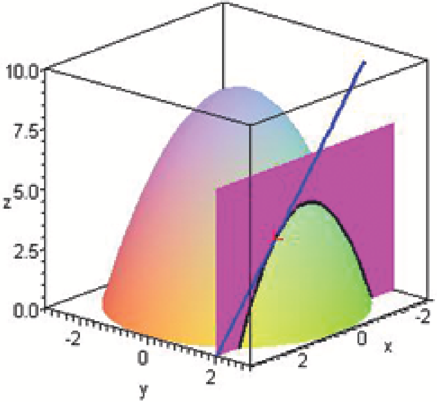


Figure: Partial derivative of $f(x, y) = 9 - x^2 - y^2$ at (1,2).

Partial derivatives in $\mathbb{R}^3$

Notations

- In $\mathbb{R}^3$ the unit coordinate vectors are denoted by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
- If $\mathbf{a} = (a, b, c)$ the partial derivatives $D_1 f(\mathbf{a})$, $D_2 f(\mathbf{a})$, and $D_3 f(\mathbf{a})$ are denoted by

$$\frac{\partial f}{\partial x}(a, b, c), \ \frac{\partial f}{\partial y}(a, b, c), \text{ and } \frac{\partial f}{\partial z}(a, b, c),$$

respectively.

Partial derivatives of second order

- Partial differentiation produces new scalar fields $D_1 f, ..., D_n f$ from a given scalar field f.
- The partial derivatives $D_1 f, ..., D_n f$ are called first order partial derivatives of f.
- For function of two variables, there are four second order partial derivatives, which are written as follows:

$$\begin{aligned} D_1(D_1 f) &= \frac{\partial^2 f}{\partial x^2}, \qquad D_2(D_2 f) = \frac{\partial^2 f}{\partial y^2} \\ D_1(D_2 f) &= \frac{\partial^2 f}{\partial x \partial y}, \quad D_2(D_1 f) = \frac{\partial^2 f}{\partial y \partial x}. \end{aligned}$$

Partial derivatives of second order
Cont...

- In the above, $D_1(D_2f)$ means the partial derivative of (D_2f) with respect to the first variable.
- We sometimes use the notation $D_{i,j}f$ for the second-order partial derivative $D_i(D_jf)$.
- For example, $D_{1,2}f = D_1(D_2f)$.

- In the ∂-notation we indicate the order of derivatives by writing

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

- This may or may not be equal to the other mixed partial derivative,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

- We shall prove later that the two mixed partials $D_1(D_2f)$ and $D_2(D_1f)$ are equal at a point if one of them is continuous in a neighborhood of the point.

Example

Consider the function

$$f(x, y) = x^2 + 5xy - 4y^2.$$

Find the second order partial derivatives of f.

Example
Solution

$$\frac{\partial f}{\partial x} = 2x + 5y \qquad \frac{\partial f}{\partial y} = 5x - 8y.$$

- A second order partial derivative should be a partial derivative of a first order partial derivative.
- So, first take two different first order partial derivatives, with respect to x or y and then, for each of those, you can take a partial derivative a second time with respect to x or y.

Example

Solution⇒Cont...

$$
\begin{aligned}
D_1(D_1 f) &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x+5y) = 2,\\
D_2(D_1 f) &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x+5y) = 5,\\
D_1(D_2 f) &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(5x-8y) = 5,\\
D_2(D_2 f) &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5x-8y) = -8.
\end{aligned}
$$

Note that f_{xy} and f_{yx} are equal in this example. While this is not always the case.

Directional Derivatives and Continuity

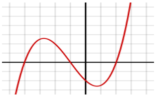
Differentiable function in one dimensional space

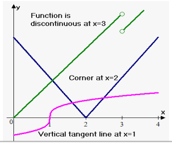
- If a is a point in the domain of a function f, then f is said to be differentiable at a if the derivative $f'(a)$ exists:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- In calculus, a differentiable function is a function whose derivative exists at each point in its domain.

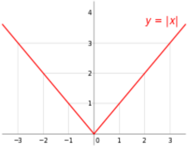
- The graph of a differentiable function must be relatively smooth, and cannot contain any breaks, bends, cusps, or any points with a vertical tangent.





Differentiability and continuity in one dimensional space

- If f is differentiable at a point a, then f must also be continuous at a.
- In particular, any differentiable function must be continuous at every point in its domain.
- The converse does not hold: a continuous function need not be differentiable.
- For example, the absolute value function is continuous at $x = 0$ but it is not differentiable at $x = 0$.



- In one-dimensional space, the existence of the derivative of a function f at a point implies continuity at that point.
- This can easily be shown by considering the definition of the derivative of a single variable function.

$$
\begin{aligned}
f(a+h) - f(a) &= \frac{f(a+h) - f(a)}{h}.h \\
\lim_{h \to 0} (f(a+h) - f(a)) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.h
\end{aligned}
$$

Differentiability and continuity in one dimensional space
Cont...

$$
\begin{aligned}
\lim_{h\to 0}\left(f(a+h)-f(a)\right) &= f'(a).0 \\
\lim_{h\to 0}\left(f(a+h)-f(a)\right) &= 0 \\
\lim_{h\to 0} f(a+h)-\lim_{h\to 0} f(a) &= 0 \\
\lim_{h\to 0} f(a+h) &= \lim_{h\to 0} f(a) \\
\lim_{h\to 0} f(a+h) &= f(a).
\end{aligned}
$$

- This shows that the existence of $f'(a)$ implies continuity of f at a.

Example 1

Check the continuity and the differentiability of the following function at $x = 0$:

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

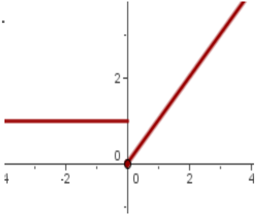
Example 1
Solution

First, we check the continuity of f at $x = 0$.

$$\lim_{x \to 0^-} 1 = 1, \quad (2)$$
$$\lim_{x \to 0^+} x = 0. \quad (3)$$

Since $(2) \neq (3)$, f is not continuous at $x = 0$.

It implies that f cannot be differentiable at $x = 0$.



Example 2

Check the continuity and the differentiability of the function $f(x) = (x-1)^{\frac{1}{3}}$ at $x = 1$.

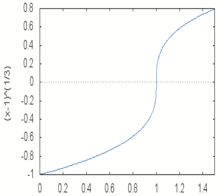
Example 2
Solution

First, we check the continuity of f at $x = 1$.

$$\lim_{x \to 1^-} (x-1)^{\frac{1}{3}} = \lim_{x \to 1^+} (x-1)^{\frac{1}{3}} = f(1) = 0.$$

So f is continuous at $x = 1$. Let's check the differentiability at $x = 1$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} \\ &= \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} \to +\infty. \end{aligned}$$



It implies that f is not differentiable at $x = 1$.

Directional derivatives and continuity in $\mathbb{R}^n$

Assume the derivative $f'(\mathbf{a};\mathbf{y})$ exists for some $\mathbf{y}$. Then if $h \neq 0$ we can write

$$\begin{aligned}
f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a}) &= \frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}.h \\
\lim_{h\to 0}\left(f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})\right) &= \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h}.h \\
\lim_{h\to 0}\left(f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})\right) &= f'(\mathbf{a};\mathbf{y}).0 \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) - \lim_{h\to 0} f(\mathbf{a}) &= 0 \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) &= \lim_{h\to 0} f(\mathbf{a}) \\
\lim_{h\to 0} f(\mathbf{a}+h\mathbf{y}) &= f(\mathbf{a}).
\end{aligned}$$

Directional derivatives and continuity in $\mathbb{R}^n$
Cont...

- This means that $f(\mathbf{x}) \to f(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$ along a straight line through $\mathbf{a}$ having direction $\mathbf{y}$.
- If $f'(\mathbf{a}; \mathbf{y})$ exists for every vector $\mathbf{y}$, then $f(\mathbf{x}) \to f(\mathbf{a})$ as $\mathbf{x} \to \mathbf{a}$ along every line through $\mathbf{a}$.
- This seems to suggest that f is continuous at $\mathbf{a}$.
- Surprisingly enough, this conclusion need not be true.

Example

Let f be the scalar field defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the above scalar field has directional derivative in every direction at $\mathbf{0}$ but which is not continuous at $\mathbf{0}$.

Example
Solution

Let $\mathbf{a} = (0,0)$ and let $\mathbf{y} = (a,b)$ be any vector. If $a \neq 0$ and $h \neq 0$ we have

$$
\begin{aligned}
\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} &= \frac{f(\mathbf{0}+h\mathbf{y})-f(\mathbf{0})}{h} \\
&= \frac{f(h\mathbf{y})-f(\mathbf{0})}{h} \\
&= \frac{f(h(a,b))-f(0,0)}{h} \\
&= \frac{f(h(a,b))}{h} \\
&= \frac{f(ha,hb)}{h} \\
&= \frac{1}{h}\left(\frac{(ha)(hb)^2}{(ha)^2+(hb)^4}\right) = \frac{ab^2}{a^2+h^2b^4}.
\end{aligned}
$$

Example

Solution ⇒ Cont...

$$\lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} = \lim_{h\to 0}\frac{ab^2}{a^2+h^2b^4}$$

$$\lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{y})-f(\mathbf{a})}{h} = \frac{ab^2}{a^2+0.b^4}.$$

$$f'(\mathbf{0};\mathbf{y}) = \frac{b^2}{a}.$$

Example

Solution $\Rightarrow$ Cont...

- If $\mathbf{y} = (0, b)$ we find, in a similar way, that $f'(\mathbf{0}; \mathbf{y}) = 0$.
- Therefore $f'(\mathbf{0}; \mathbf{y})$ exists for all directions $\mathbf{y}$.
- Also, $f(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$ along any straight line through the origin.
- However, at each point of the parabola $x = y^2$ (except at the origin) the function f has the value $1/2$.

Example

Solution ⇒ Cont...

- Since such points exist arbitrarily close to the origin and since $f(\mathbf{0}) = 0$, the function f is not continuous at $\mathbf{0}$.
- The above example describes a scalar field which has a directional derivative in every direction at $\mathbf{0}$ but which is not continuous at $\mathbf{0}$.

Remark

- The above example shows that the existence of all directional derivatives at a point fails to imply continuity at that point.
- For this reason, directional derivatives are somewhat unsatisfactory extension of the one-dimensional concept of derivative.
- A more suitable generalization exists which implies continuity and, at the same time, permits us to extend the principal theorems of one dimensional derivative theory to the higher dimensional case.
- This is called the **total derivative**.

Total Derivative

What is total derivative?

- In the previous section, we discussed partial derivatives, which represent the instantaneous rates of change of a function, f, with respect to a single variable, while keeping all of the other independent variables constant.

- We can think of each partial derivative as the instantaneous rate of change of f, at a point $\mathbf{a}$, as the point moves in a direction parallel to the corresponding coordinate axis.

- Another way to say this is that the partial derivative, with respect to x_i is the instantaneous rate of change of f, at a point $\mathbf{a}$, as the point moves in the direction of the corresponding standard basis vector, $\mathbf{e}_i$.

- This naturally leads us to look at the instantaneous rates of change of f, at a point $\mathbf{a}$, as the point moves in an arbitrary direction, with an arbitrary speed, i.e., as the point moves with an arbitrary velocity $\mathbf{v}$.

- Thus, we define the total derivative of f, at $\mathbf{a}$, not as a number, but rather as a function which returns a number for each specified velocity vector.

- How your calculator gives answer for $\sin x$ for any particular value of x that you request?
- It can not remember sin value for every x, because this requires more memory.
- So it uses a polynomial approximation for that.

$$
\begin{aligned}
f'(a) &\approx \frac{f(x)-f(a)}{(x-a)} \\
f(x) &\approx f(a)+f'(a)(x-a) \\
\text{For example } x &= 0.2 \Rightarrow \\
\sin(0.2) &\approx \sin 0+\cos 0(0.2-0) \\
&\approx 0.2
\end{aligned}
$$

- We can obtain a better result using higher order Taylor polynomials.

Approximating a differentiable function by a Taylor polynomial

We recall that in the one-dimensional case a function f with a derivative at a can be approximated near a by a linear Taylor polynomial. If $f'(a)$ exists we let $E(a, h)$ denote the difference

$$E(a,h) = \begin{cases} \dfrac{f(a+h) - f(a)}{h} - f'(a) & \text{if } h \neq 0, \\ 0 & \text{if } h = 0. \end{cases} \tag{4}$$

Approximating a differentiable function by a Taylor polynomial
Cont...

- From (4) we obtain the formula;

$$f(a+h) = f(a) + f'(a)h + hE(a,h),$$

  an equation which holds also for $h = 0$.

- This is the first-order Taylor formula for approximating $f(a+h) - f(a)$ by $f'(a)h$.
- The error committed is $hE(a,h)$.
- From (4) we see that $E(a,h) \to 0$ as $h \to 0$.
- Therefore the error $hE(a,h)$ is of smaller order than h for small h.

- This property of approximating a differentiable function by a linear function suggests a way of extending the concept of differentiability to the higher-dimensional case.
- Let $f : \mathbf{S} \to \mathbb{R}$ be a scalar field defined on a set $\mathbf{S}$ in $\mathbb{R}^n$.
- Let $\mathbf{a}$ be an interior point of $\mathbf{S}$, and let $\mathbf{B}(\mathbf{a}; r)$ be an n-ball lying in $\mathbf{S}$.
- Let $\mathbf{v}$ be a vector with $\|\mathbf{v}\| < r$, so that $\mathbf{a} + \mathbf{v} \in \mathbf{B}(\mathbf{a}; r)$.

Definition of a differentiable scalar field

We say that f is differentiable at $\mathbf{a}$ if there exists a linear transformation

$$T_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$$

from $\mathbb{R}^n$ to $\mathbb{R}$, and a scalar function $E(\mathbf{a}, \mathbf{v})$ such that

$$f(\mathbf{a} + \mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| E(\mathbf{a}, \mathbf{v}), \tag{5}$$

for $\|\mathbf{v}\| < r$, where $E(\mathbf{a}, \mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$. The linear transformation $T_{\mathbf{a}}$ is called the total derivative of f at $\mathbf{a}$.

Definition of a differentiable scalar field
Cont...

- The total derivative was introduced by W.H. Young in 1908 and by M. Frechet in 1911 in more general context.
- The total derivative $T_{\mathbf{a}}$ is a linear transformation, not a number.
- The function value $T_{\mathbf{a}}(\mathbf{v})$ is a real number; it is defined for every point $\mathbf{v}$ in $\mathbb{R}^n$.

Definition of a differentiable scalar field

Cont...

- The equation (5), which holds for $\|\mathbf{v}\| < r$, is called a first-order Taylor formula for $f(\mathbf{a} + \mathbf{v})$.

- It gives a linear approximation, $T_{\mathbf{a}}(\mathbf{v})$, to the difference $f(\mathbf{a} + \mathbf{v}) - f(\mathbf{a})$.

- The error in the approximation is $\|\mathbf{v}\| E(\mathbf{a}, \mathbf{v})$, a term which is of smaller order than $\|\mathbf{v}\|$ as $\|\mathbf{v}\| \to 0$; that is, $E(\mathbf{a}, \mathbf{v}) = O(\|\mathbf{v}\|)$ as $\|\mathbf{v}\| \to 0$.

Theorem (3.2)

Assume f is differentiable at $\mathbf{a}$ with total derivative $T_{\mathbf{a}}$. Then the derivative $f'(\mathbf{a};\mathbf{y})$ exists for every $\mathbf{y}$ in $\mathbb{R}^n$ and we have

$$T_{\mathbf{a}}(\mathbf{y}) = f'(\mathbf{a};\mathbf{y}). \tag{6}$$

Moreover, $f'(\mathbf{a};\mathbf{y})$ is a linear combination of the components of $\mathbf{y}$. In fact, if $\mathbf{y} = (y_1, ..., y_n)$, we have

$$f'(\mathbf{a};\mathbf{y}) = \sum_{k=1}^{n} D_k f(\mathbf{a}) y_k. \tag{7}$$

Theorem (3.2)
Proof

The equation (6) holds trivially if $\mathbf{y} = \mathbf{0}$ since both $T_{\mathbf{a}}(\mathbf{0}) = 0$ and $f'(\mathbf{a}; \mathbf{0}) = 0$.

Therefore we can assume that $\mathbf{y} \neq \mathbf{0}$.

Since f is differentiable at $\mathbf{a}$ we have a Taylor formula,

$$f(\mathbf{a} + \mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| E(\mathbf{a}, \mathbf{v}), \tag{8}$$

for $\|\mathbf{v}\| < r$ for some $r > 0$, where $E(\mathbf{a}, \mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$.

In this formula we take $\mathbf{v} = h\mathbf{y}$, where $h \neq 0$ and $|h|\|\mathbf{y}\| < r$.

Then $\|\mathbf{v}\| < r$.

Since $T_{\mathbf{a}}$ is linear we have $T_{\mathbf{a}}(\mathbf{v}) = T_{\mathbf{a}}(h\mathbf{y}) = hT_{\mathbf{a}}(\mathbf{y})$.

Therefore (8) gives us

$$\begin{aligned}
f(\mathbf{a}+\mathbf{v}) &= f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| E(\mathbf{a},\mathbf{v}) \\
f(\mathbf{a}+h\mathbf{y}) &= f(\mathbf{a}) + hT_{\mathbf{a}}(\mathbf{y}) + |h|\|\mathbf{y}\| E(\mathbf{a},\mathbf{v}) \\
f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a}) &= hT_{\mathbf{a}}(\mathbf{y}) + |h|\|\mathbf{y}\| E(\mathbf{a},\mathbf{v}) \\
\frac{f(\mathbf{a}+h\mathbf{y}) - f(\mathbf{a})}{h} &= T_{\mathbf{a}}(\mathbf{y}) + \frac{|h|\|\mathbf{y}\|}{h} E(\mathbf{a},\mathbf{v}). \qquad (9)
\end{aligned}$$

- Since $\|\mathbf{v}\| \to 0$ as $h \to 0$ and since $|h|/h = \pm 1$, the right hand member of (9) tends to the limit $T_{\mathbf{a}}(\mathbf{y})$ as $h \to 0$.
- Therefore the left-hand member tends to the same limit.
- This proves (6).

Now we use the linearity of $T_{\mathbf{a}}$ to deduce (7). If $\mathbf{y} = (y_1, ..., y_n)$ we have $\mathbf{y} = \sum_{k=1}^{n} y_k \mathbf{e}_k$, hence

$$\begin{aligned} T_{\mathbf{a}}(\mathbf{y}) &= T_{\mathbf{a}}\left(\sum_{k=1}^{n} y_k \mathbf{e}_k\right) \\ &= \sum_{k=1}^{n} y_k T_{\mathbf{a}}(\mathbf{e}_k) \\ &= \sum_{k=1}^{n} y_k f'(\mathbf{a}; \mathbf{e}_k) \\ &= \sum_{k=1}^{n} y_k D_k f(\mathbf{a}). \end{aligned}$$

The Gradient of a Scalar Field

What is gradient of a scalar field?

- Assume that there is a heat source in a room and the temperature does not change over time.
- Suppose the temperature in that room is given by a scalar field, f, so at each point (x, y, z) the temperature is $f(x, y, z)$.
- At each point in the room, the gradient of f at that point will show the direction the temperature rises most quickly.
- The magnitude of the gradient will determine how fast the temperature rises in that direction.

What is gradient of a scalar field?

The gradient of the function $f(x, y) = -(\cos^2 x + \cos^2 y)^2$

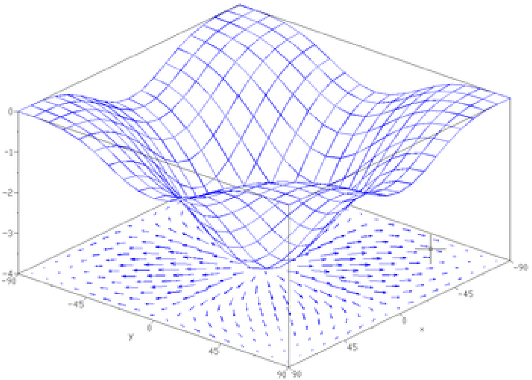


Figure: The gradient of the function $f(x, y) = -(\cos^2 x + \cos^2 y)^2$ depicted as a projected vector field on the bottom plane

Mathematical aspect of the gradient of a scalar field

- The gradient of a scalar field is a vector field that points in the direction of the greatest rate of increase of the scalar field.
- The magnitude is the rate of change and which points in the direction of the greatest rate of increase of the scalar field.
- If the vector is resolved, its components represent the rate of change of the scalar field with respect to each directional component.

Mathematical aspect of the gradient of a scalar field
Notations

- The gradient of a scalar field f is denoted ∇f.
- Where ∇ denotes the vector differential operator, del.
- The notation "grad(f)" is also commonly used for the gradient.

- Hence for a two-dimensional scalar field $f(x, y)$,

$$\operatorname{grad} f(x,y) = \nabla f(x,y) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

- And for a three-dimensional scalar field $f(x, y, z)$,

$$\begin{aligned} \operatorname{grad} f(x,y,z) = \nabla f(x,y,z) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right). \end{aligned}$$

- For a n-dimensional scalar field $f(x_1, x_2, ..., x_n)$,

$$\begin{aligned}
\operatorname{grad} f(x_1, x_2, ..., x_n) = \nabla f &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right) f \\
&= \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\right).
\end{aligned}$$

In 2-space the gradient vector is often written as

$$\nabla f(x, y) = \frac{\partial f(x, y)}{\partial x}\mathbf{i} + \frac{\partial f(x, y)}{\partial y}\mathbf{j}.$$

In 3-space the corresponding formula is

$$\nabla f(x, y, z) = \frac{\partial f(x, y, z)}{\partial x}\mathbf{i} + \frac{\partial f(x, y, z)}{\partial y}\mathbf{j} + \frac{\partial f(x, y, z)}{\partial z}\mathbf{k}.$$

In n-space the corresponding formula is

$$\begin{aligned}\nabla f(x_1, x_2, ..., x_n) &= \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_1}\mathbf{e}_1 + \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_2}\mathbf{e}_2 \\ &\quad + ... + \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_n}\mathbf{e}_n.\end{aligned}$$

Examples

For following scalar fields, calculate ∇f:

1. $f(x, y) = 8x + 5y$.
2. $f(x, y, z) = x^4 yz$.
3. $f(x, y) = x^2 \sin 5y$.

Given scalar field $f(x, y) = 8x + 5y$:

$$\begin{aligned} \nabla f(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= (8, 5). \end{aligned}$$

Given scalar field $f(x, y) = x^4yz$:

$$\begin{aligned} \nabla f(x, y, z) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(4x^3yz, x^4z, x^4y\right). \end{aligned}$$

Given scalar field $f(x, y) = x^2 \sin 5y$:

$$\begin{aligned} \nabla f(x, y) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= (2x \sin(5y), 5x^2 \cos(5y)). \end{aligned}$$

The first order Taylor formula using gradient

The formula in Theorem (3.2), which expresses $f'(\mathbf{a};\mathbf{y})$ as a linear combination of the components of $\mathbf{y}$, can now be written as a dot product,

$$f'(\mathbf{a};\mathbf{y}) = \sum_{k=1}^{n} D_k f(\mathbf{a}) y_k = \nabla f(\mathbf{a}).\mathbf{y}, \tag{10}$$

where $\nabla f(\mathbf{a})$ is the gradient of the scalar field f.

The first order Taylor formula using gradient
Cont...

- If f is a differentiable function at point $\mathbf{a}$ we have a Taylor formula,

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + \|\mathbf{v}\| E(\mathbf{a},\mathbf{v}).$$

- From Theorem (3.2) we have

$$\begin{aligned} T_{\mathbf{a}}(\mathbf{y}) &= f'(\mathbf{a};\mathbf{y}) \\ T_{\mathbf{a}}(\mathbf{v}) &= f'(\mathbf{a};\mathbf{v}) = \nabla f(\mathbf{a}).\mathbf{v} \text{ (From (10))}. \end{aligned}$$

- The first order Taylor formula can now be written in the form

$$f(\mathbf{a}+\mathbf{v}) = f(\mathbf{a}) + \nabla f(\mathbf{a}).\mathbf{v} + \|\mathbf{v}\| E(\mathbf{a},\mathbf{v}), \tag{11}$$

where $E(\mathbf{a},\mathbf{v}) \to 0$ as $\|\mathbf{v}\| \to 0$.

The first order Taylor formula using gradient
Cont...

- The above form of Taylor formula resembles the one-dimensional Taylor formula, with the gradient vector $\nabla f(\mathbf{a})$ playing the role of the derivative $f'(\mathbf{a})$.
- From the Taylor formula we can easily prove that differentiability implies continuity.

Theorem (3.3)

If a scalar field f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

Theorem (3.3)
Proof

From equation (11) we have

$$f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a}) \;=\; \nabla f(\mathbf{a}).\mathbf{v} + \|\mathbf{v}\|E(\mathbf{a},\mathbf{v})$$

By taking modulus from both side we have

$$|f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| \;=\; |\nabla f(\mathbf{a}).\mathbf{v} + \|\mathbf{v}\|E(\mathbf{a},\mathbf{v})|.$$

Applying the triangle inequality we find

$$0 \leq |f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| \;\leq\; |\nabla f(\mathbf{a}).\mathbf{v}| + |\|\mathbf{v}\|E(\mathbf{a},\mathbf{v})|.$$

Applying the Cauchy-Schwarz inequality we find

$$0 \leq |f(\mathbf{a}+\mathbf{v}) - f(\mathbf{a})| \leq \|\nabla f(\mathbf{a})\|\|\mathbf{v}\| + \|\mathbf{v}\||E(\mathbf{a},\mathbf{v})|.$$

This shows that $f(\mathbf{a}+\mathbf{v}) \to f(\mathbf{a})$ as $\|\mathbf{v}\| \to 0$, so f is continuous at $\mathbf{a}$.

Example

Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by,

$$g(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & , \text{ if } (x, y) \neq (0, 0), \\ 0 & , \text{ if } (x, y) = (0, 0). \end{cases}$$

(i) Using the definition, show that $\dfrac{\partial}{\partial x} g(0, 0) = 0$ and $\dfrac{\partial}{\partial y} g(0, 0) = 0$.

(ii) Check the continuity of g at (0, 0).

(iii) Check the differentiability of g at (0, 0).

(iv) What conclusions can be obtained from above results on the differentiability of scalar fields and their partial derivatives at some points?

Example
Solution

(i)

$$
\begin{aligned}
\frac{\partial}{\partial x} g(x,y) &= \lim_{h\to 0} \frac{g(x+h,y)-g(x,y)}{h} \\
\frac{\partial}{\partial x} g(0,0) &= \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{h.0}{h^2+0}-0\right) \\
&= 0.
\end{aligned}
$$

Example
Solution

$$\begin{aligned}
\frac{\partial}{\partial y} g(x,y) &= \lim_{h\to 0} \frac{g(x,y+h)-g(x,y)}{h} \\
\frac{\partial}{\partial y} g(0,0) &= \lim_{h\to 0} \frac{g(0,0+h)-g(0,0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{0.h}{0+h^2}-0\right) \\
&= 0.
\end{aligned}$$

(ii) Consider the limit of the function $g(x, y)$ when $(x, y) \to (0, 0)$ along the path $y = mx$, where $m \in \mathbb{R}$. Then we have

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{x\to 0} g(x, mx) = \lim_{x\to 0} \frac{x.mx}{x^2 + (mx)^2} = \frac{m}{1+m^2}.$$

This limit changes when m changes. That is limit is not unique. Therefore $\lim_{(x,y)\to(0,0)} g(x, y)$ does not exists. It implies that g is not continuous at $(0, 0)$.

(iii) Since g is not continuous at $(0,0)$, g is not differentiable at $(0,0)$.

(iv) There exists some scalar fields which are not differentiable at a point but they have partial derivatives at that point.

Sufficient Conditions for Differentiability

Motivating example

- Consider the function

$$g(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

  discussed in the previous Section.

- For this function, both partial derivatives $D_1 g(\mathbf{0})$ and $D_2 g(\mathbf{0})$ exist.

- But g is not continuous at $\mathbf{0}$, hence g cannot be differentiable at $\mathbf{0}$.

Remark

- If f is differentiable at $\mathbf{a}$, then all partial derivatives $D_1 f(\mathbf{a}), ..., D_n f(\mathbf{a})$ exist.
- However, the existence of all these partial derivatives does not necessarily imply that f is differentiable at $\mathbf{a}$.

Theorem (3.4)
A sufficient condition for differentiability

Assume that the partial derivatives $D_1 f, ..., D_n f$ exist in some n-ball $B(\mathbf{a})$ and are continuous at $\mathbf{a}$. Then f is differentiable at $\mathbf{a}$.

Note: Sufficient Conditions If we say that "x is a sufficient condition for y," then we mean that if we have x, we know that y must follow. In other words, x guarantees y.

- The above theorem shows that the existence of continuous partial derivatives at a point implies differentiability at that point.
- A scalar field satisfying the hypothesis of **Theorem 3.4** is said to be continuously differentiable.

A differentiable function with discontinuous partial derivatives

- The **Theorem 3.4** states that continuous partial derivatives are sufficient for a function to be differentiable.
- But the converse of the **Theorem 3.4** is not true.
- That means, it is possible for a differentiable function to have discontinuous partial derivatives.

Example 1

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real valued function defined such that

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & ; \text{ if } x \neq 0, \\ 0 & ; \text{ if } x = 0. \end{cases}$$

(a) Evaluate $f_x(x,y)$ and $f_y(x,y)$.

(b) Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(x,y) = (0,0)$.

(c) What can you say about differentiability of f at the point $(0,0)$?

Example 1
Solution

(a)

$$
\begin{aligned}
f_x(x,y) &= \frac{(x^2+y^4)y^2 - xy^2(2x)}{(x^2+y^4)^2} \\
&= \frac{y^6 - x^2y^2}{(x^2+y^4)^2} \\
f_y(x,y) &= \frac{(x^2+y^4)2xy - xy^2 4y^3}{(x^2+y^4)^2} \\
&= \frac{2x^3y - 2xy^5}{(x^2+y^4)^2}.
\end{aligned}
$$

(b) Consider the limit of $f_x(x,y)$ when $(x,y) \to (0,0)$ along the path $y = mx$, where $m \in \mathbb{R}$. Then we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f_x(x,y) &= \lim_{x\to 0} f_x(x,mx) \\
&= \lim_{x\to 0} \frac{(mx)^6 - x^2(mx)^2}{(x^2+(mx)^4)^2} \\
&= \lim_{x\to 0} \frac{mx^6 - m^2x^4}{x^4(1+m^4x^2)^2} \\
&= -m^2.
\end{aligned}$$

This limit depends on m. That is limit is not unique. Therefore the limit of $f_x(x,y)$ when $(x,y) \to (0,0)$ does not exist. So, $f_x(x,y)$ is not continuous at $(x,y) = (0,0)$.

Example 1
Solution

Consider the limit of $f_y(x, y)$ when $(x, y) \to (0, 0)$ along the path $y = bx$, where $b \in \mathbb{R}$. Then we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f_y(x, y) &= \lim_{x\to 0} f_y(x, bx) \\
&= \lim_{x\to 0} \frac{2x^3(bx) - 2x(bx)^5}{(x^2 + (bx)^4)^2} \\
&= \lim_{x\to 0} \frac{2bx^4 - 2b^5x^6}{x^4(1 + b^4x^2)^2} \\
&= 2b.
\end{aligned}$$

This limit also depends on b. That is limit is not unique. Therefore the limit of $f_y(x, y)$ when $(x, y) \to (0, 0)$ does not exist. So, $f_y(x, y)$ is not continuous at $(x, y) = (0, 0)$.

(c) A function can be differentiable even with discontinuous partial derivatives. So, based on the fact that $f_x(x, y)$ and $f_y(x, y)$ are discontinuous, we cannot make any conclusion about the differentiable of $f(x, y)$ at $(0, 0)$.

But we can show that $f(x, y)$ is not continuous at (0,0) (Try as an Exercise). Since $f(x, y)$ is not continuous at (0, 0), it cannot be differentiable at (0, 0).

Example 2

Consider the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Although the function is differentiable, its partial derivatives oscillate wildly near the origin, creating a discontinuity there.

It provides a counter example showing that partial derivatives do not need to be continuous for a function to be differentiable, demonstrating that the converse of the **Theorem 3.4** is not true.

Sufficient Conditions for the Equality of Mixed Partial Derivatives

Mixed partial derivatives

- If f is a real-valued function of two variables, the two mixed partial derivatives $D_{1,2}f$ and $D_{2,1}f$ are not necessarily equal.
- By $D_{1,2}f$ we mean $D_1(D_2 f) = \dfrac{\partial^2 f}{\partial x \partial y}$, and by $D_{2,1}f$ we mean $D_2(D_1 f) = \dfrac{\partial^2 f}{\partial y \partial x}$.

Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real valued function defined such that

$$f(x,y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & ; \text{ if } (x,y) \neq (0,0), \\ 0 & ; \text{ if } (x,y) = (0,0). \end{cases}$$

Determine $D_{2,1}f(0,0)$ and $D_{1,2}f(0,0)$.

The defintion of $D_{2,1}f(0,0)$ states that

$$D_{2,1}f(0,0) = \lim_{k\to 0} \frac{D_1 f(0,k) - D_1 f(0,0)}{k}. \tag{12}$$

Now we have

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

and, if $(x,y) \neq (0,0)$, we find

$$D_1 f(x,y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2}.$$

Example
Solution⇒Cont...

Therefore, if $k \neq 0$ we have $D_1 f(0,k) = -k^5/k^4 = -k$ and hence

$$\frac{D_1 f(0,k) - D_1 f(0,0)}{k} = -1.$$

Using this in (12) we find that $D_{2,1} f(0,0) = -1$.

A similar argument shows that $D_{1,2} f(0,0) = 1$, and hence $D_{2,1} f(0,0) \neq D_{1,2} f(0,0)$.

Remark

- In the example just treated the two mixed partials $D_{1,2}f$ and $D_{2,1}f$ are not both continuous at the origin.
- It can be shown that the two mixed partials are equal at a point (a, b) if at least one of them is continuous in a neighborhood of the point.

Theorem (3.5)

Sufficient conditions for the equality of mixed partial derivatives

Assume f is a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, $D_{1,2} f$ and $D_{2,1} f$ exist on an open set $\mathbf{S}$. If (a, b) is a point in $\mathbf{S}$ at which both $D_{1,2} f$ and $D_{2,1} f$ are continuous, we have

$$D_{1,2} f(a, b) = D_{2,1} f(a, b). \tag{13}$$

Theorem (3.6)

Let f be a scalar field such that the partial derivatives $D_1 f$, $D_2 f$, and $D_{2,1} f$ exist on an open set $\mathbf{S}$ containing (a, b). Assume further that $D_{2,1} f$ is continuous on $\mathbf{S}$. Then the derivative $D_{1,2} f(a, b)$ exists and we have

$$D_{1,2} f(a, b) = D_{2,1} f(a, b). \tag{14}$$

The Relationship between Directional Derivative and Gradient Vector

Directional derivative and gradient vector

- When $\mathbf{y}$ is a unit vector, the directional derivative $f'(\mathbf{a};\mathbf{y})$ has a simple geometric relation to the gradient vector.
- Assume that $\nabla f(\mathbf{a}) \neq \mathbf{0}$ and let θ denote the angle between $\mathbf{y}$ and $\nabla f(\mathbf{a})$.
- Then we have

$$f'(\mathbf{a};\mathbf{y}) = \nabla f(\mathbf{a}).\mathbf{y} = \|\nabla f(\mathbf{a})\|\|\mathbf{y}\|\cos\theta = \|\nabla f(\mathbf{a})\|\cos\theta.$$

- This shows that the directional derivative is simply the component of the gradient vector in the direction of **y**.
- The derivative is largest when $\cos\theta = 1$, that is, when **y** has the same direction as $\nabla f(\mathbf{a})$.

- In other words, at a given point $\mathbf{a}$, the scalar field undergoes its maximum rate of change in the direction of the gradient vector.
- Moreover, this maximum is equal to the length of the gradient vector.
- When $\nabla f(\mathbf{a})$ is orthogonal to $\mathbf{y}$, the directional derivative $f'(\mathbf{a};\mathbf{y})$ is 0.

Example 1

What is the directional derivative of the function $f(x, y) = 4x^2 + y^2$ at the point $x = 2$ and $y = 2$ in the direction of the vector $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$.

Example 1
Solution

The gradient is $\nabla f(x, y) = 8x\mathbf{i} + 2y\mathbf{j}$, which is at the point $(2, 2)$ is $\nabla f(2, 2) = 16\mathbf{i} + 4\mathbf{j}$.

The direction is given by $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$.

The unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$ is $\dfrac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}}$. Hence,

$$\begin{aligned} f'(\mathbf{a}; \hat{\mathbf{u}}) &= \nabla f(\mathbf{a}).\hat{\mathbf{u}} \\ &= \nabla f(2, 2).\hat{\mathbf{u}} \\ &= (16\mathbf{i} + 4\mathbf{j}) \cdot \frac{2\mathbf{i}+\mathbf{j}}{\sqrt{5}} \\ &= \frac{36}{\sqrt{5}} \end{aligned}$$

Example 2

Find the direction in which the function

$$f(x, y) = \sin x + e^{y-1}$$

has the greatest rate of change at the point (0, 1).

At a given point, a scalar field undergoes its maximum rate of change in the direction of the gradient vector.

The gradient is $\nabla f(x, y) = \cos x \, \mathbf{i} + e^{y-1} \, \mathbf{j}$.

Thus, the gradient vector at (0, 1) is equal to
$\nabla f(0, 1) = \cos 0 \, \mathbf{i} + e^{1-1} \, \mathbf{j} = \mathbf{i} + \mathbf{j}$.

Exercise

Find the directional derivative of

$$f(x, y) = \frac{1}{1 + x^2 + y^2},$$

at the point (1, 0) in the direction of the vector $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

Answer is $-\frac{2}{5}$

A Chain Rule for Derivatives of Scalar Fields

A function of a function

- Consider the expression $\sin t^2$.
- It is clear that this is different from the straightforward sine function, $\sin t$.
- We are finding the sine of t^2, not simply the sine of t.
- We call such an expression a "function of a function" or a "composite function".

A function of a function
Cont...

- Suppose, in general, that we have two functions, $f(t)$ and $r(t)$.
- Then $g(t) = f[r(t)]$ is a function of a function.
- In our case, the function f is the sine function and the function r is the square function.
- We could identify them more mathematically by saying that $f(t) = \sin t$ and $r(t) = t^2$, so that $f[r(t)] = f(t^2) = \sin t^2$.

The chain rule in one-dimensional space

In one-dimensional derivative theory, the chain rule enables us to compute the derivative of a function of a function $g(t) = f[r(t)]$ by the formula

$$g'(t) = f'[r(t)].r'(t).$$

(i) $y = \sin x^2$

(ii) $y = (2x - 3)^{12}$

(iii) $y = e^{x^3}$

(iv) $y = e^{1+x^2}$

(v) $y = \sin(x + e^x)$

The chain rule for derivatives of scalar fields

This Section provides an extension of the formula when f is replaced by a scalar field defined on a set in n-space and r is replaced by a vector-valued function of a real variable with values in the domain of f.

The chain rule for derivatives of scalar fields
Cont...

- It is easy to conceive of examples in which the composition of a scalar field and a vector field might arise.
- For instance, suppose $f(\mathbf{x})$ measures the temperature at a point $\mathbf{x}$ of a solid in 3-space, and suppose we wish to know how the temperature changes as the point $\mathbf{x}$ varies along a curve C lying in the solid.
- If the curve is described by a vector-valued function $\mathbf{r}$ defined on an interval $[a, b]$, we can introduce a new function g by means of the formula

$$g(t) = f[\mathbf{r}(t)] \quad \text{if } a \leq t \leq b.$$

The chain rule for derivatives of scalar fields
Cont...

- This composite function g expresses the temperature as a function of the parameter t, and its derivative $g'(t)$ measures the rate of chage of the temperature along the curve.
- The following extension of the chain rule enables us to compute the derivative $g'(t)$ without determining $g(t)$ explicitly.

Theorem 3.7
Chain rule

Let f be a scalar field defined on an open set $\mathbf{S}$ in $\mathbb{R}^n$, and let $\mathbf{r}$ be a vector-valued function which maps an interval $\mathbb{J}$ from $\mathbb{R}^1$ into $\mathbf{S}$. Define the composite function $g = f \circ \mathbf{r}$ on $\mathbb{J}$ by the equation

$$g(t) = f[\mathbf{r}(t)] \quad \text{if } t \in \mathbb{J}.$$

Let t be a point in $\mathbb{J}$ at which $\mathbf{r}'(t)$ exists and assume that f is differentiable at $\mathbf{r}(t)$. Then $g'(t)$ exists and is equal to the dot product

$$g'(t) = \nabla f(\mathbf{a}).\mathbf{r}'(t), \text{ where } \mathbf{a} = \mathbf{r}(t). \tag{15}$$

Let $\mathbf{a} = \mathbf{r}(t)$, where t is a point in $\mathbb{J}$ at which $\mathbf{r}'(t)$ exists.

Since $\mathbf{S}$ is open there is an n-ball $\mathbf{B}(\mathbf{a})$ lying in $\mathbf{S}$.

We take $h \neq 0$ but small enough so that $\mathbf{r}(t+h)$ lies in $\mathbf{B}(\mathbf{a})$, and we let $\mathbf{y} = \mathbf{r}(t+h) - \mathbf{r}(t)$.

Note that $\mathbf{y} \to \mathbf{0}$ as $h \to 0$.

Now we have

$$\begin{aligned} g(t+h) - g(t) &= f[\mathbf{r}(t+h)] - f[\mathbf{r}(t)] \\ &= f(\mathbf{a}+\mathbf{y}) - f(\mathbf{a}). \end{aligned} \tag{16}$$

Applying the first-order Taylor formula for f we have

$$f(\mathbf{a}+\mathbf{y})-f(\mathbf{a})=\nabla f(\mathbf{a}).\mathbf{y}+\|\mathbf{y}\|E(\mathbf{a},\mathbf{y}), \tag{17}$$

where $E(\mathbf{a},\mathbf{y})\to 0$ as $\|\mathbf{y}\|\to 0$.

From (16) and (17) we have

$$g(t+h)-g(t)=\nabla f(\mathbf{a}).\mathbf{y}+\|\mathbf{y}\|E(\mathbf{a},\mathbf{y}).$$

Since $\mathbf{y}=\mathbf{r}(t+h)-\mathbf{r}(t)$ this gives us

$$\begin{aligned}\frac{g(t+h)-g(t)}{h} &= \nabla f(\mathbf{a}).\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}\\ &\quad+\frac{\|\mathbf{r}(t+h)-\mathbf{r}(t)\|}{h}E(\mathbf{a},\mathbf{y})\end{aligned}$$

By letting $h \to 0$ we obtain:

$$\begin{aligned}
\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} &= \lim_{h\to 0}\nabla f(\mathbf{a}).\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \\
&\quad + \lim_{h\to 0}\frac{\|\mathbf{r}(t+h)-\mathbf{r}(t)\|}{h}E(\mathbf{a},\mathbf{y}) \\
\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} &= \nabla f(\mathbf{a}).\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}+0 \\
g'(t) &= \nabla f(\mathbf{a}).\mathbf{r}'(t).
\end{aligned}$$

- When the function $\mathbf{r}$ describes a curve C, the derivative $\mathbf{r}'$ is the velocity vector (tangent to the curve) and derivative g' in Equation (15) is the derivative of f with respect to the velocity vector, assuming that $\mathbf{r}' \neq \mathbf{0}$.

- If $\mathbf{T}(t)$ is a unit vector in the direction of $\mathbf{r}'(t)$ ($\mathbf{T}$ is the unit tangent vector), the dot product $\nabla f[\mathbf{r}(t)].\mathbf{T}(t)$ is called the directional derivative of f along the curve C or in the direction of C.

- For a plane curve we can write

$$\mathbf{T}(t) = \cos\alpha(t)\mathbf{i} + \cos\beta(t)\mathbf{j},$$

where $\alpha(t)$ and $\beta(t)$ are the angles made by the vector $\mathbf{T}(t)$ and the positive x- and y-axes; the directional derivative of f along C becomes

$$\nabla f[\mathbf{r}(t)].\mathbf{T}(t) = D_1 f[\mathbf{r}(t)]\cos\alpha(t) + D_2 f[\mathbf{r}(t)]\cos\beta(t).$$

- This formula is often written more briefly as

$$\nabla f.\mathbf{T} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta.$$

- Since the directional derivative along C is defined in terms of **T**, its value depends on the parametric representation chosen for C.

- A change of the representation could reverse the direction of **T**; this in turn, would reverse the sign of the directional derivative.

Example 2

Find the directional derivative of the scalar field $f(x, y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point (1,2).

Example 2
Cont...

At an arbitary point (x, y) the gradient vector is

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\ &= (2x-3y)\mathbf{i} - 3x\mathbf{j}.\end{aligned}$$

At the point (1,2) we have $\nabla f(1,2) = -4\mathbf{i} - 3\mathbf{j}$.

The parabola can be represented parametrically by the vector equation $\mathbf{r}(t) = t\mathbf{i} + (t^2 - t + 2)\mathbf{j}$.

Therefore $\mathbf{r}(1) = \mathbf{i} + 2\mathbf{j}$, $\mathbf{r}'(t) = \mathbf{i} + (2t-1)\mathbf{j}$, and $\mathbf{r}'(1) = \mathbf{i} + \mathbf{j}$.

For this representation of C the unit tangent vector $\mathbf{T}(1)$ is $(\mathbf{i} + \mathbf{j})/\sqrt{2}$ and the required directional derivative is $\nabla f(1,2).\mathbf{T}(1) = -7/\sqrt{2}$.

Example 3

Let f be nonconstant scalar field, differentiable everywhere in the plane, and let c be a constant. Assume the Cartesian equation $f(x, y) = c$ describes a curve C having a tangent at each of its points. Prove that f has the following properties at each point of C:

(a) The gradient vector ∇f is normal to C.

(b) The directional derivative of f is zero along C.

(c) The directional derivative of f has its largest value in a direction normal to C.

Example 3
Cont...

If $\mathbf{T}$ is a unit tangent vector to C, the directional derivative of f along C is the dot product $\nabla f.\mathbf{T}$.

This product is zero if ∇f is perpendicular to $\mathbf{T}$, and it has its largest value if ∇f is parallel to $\mathbf{T}$.

Therefore both statements (b) and (c) are consequences of (a).

To prove (a), consider any plane curve Γ with a vector equation of the form $\mathbf{r}(t) = X(t)\mathbf{i} + Y(t)\mathbf{j}$ and introduce the function $g(t) = f[\mathbf{r}(t)]$.

Example 3
Cont...

By the chain rule we have $g'(t) = \nabla f[\mathbf{r}(t)].\mathbf{r}'(t)$.

When $\Gamma = C$, the function g has the constant value c so $g'(t) = 0$ if $\mathbf{r}(t) \in C$.

Since $g' = \nabla f.\mathbf{r}'$, this shows that ∇f is perpendicular to $\mathbf{r}'$ on C; hence ∇f is normal to C.

Level sets

Let f be a scalar field defined on a set $\mathbf{S}$ in $\mathbb{R}^n$ and consider those points $\mathbf{x}$ in $\mathbf{S}$ for which $f(\mathbf{x})$ has a constant value, say $f(\mathbf{x}) = c$. Denote this set by $L(c)$, so that

$$L(c) = \{\mathbf{x} | \mathbf{x} \in \mathbf{S} \text{ and } f(\mathbf{x}) = c\}.$$

The set $L(c)$ is called a level set of f. In $\mathbb{R}^2$, $L(c)$ is called a level curve; in $\mathbb{R}^3$, it is called a level surface.

- A level curve of a function $f(x, y)$ is the curve of points (x, y) where $f(x, y)$ is some constant value.
- A level curve is simply a cross section of the graph of $z = f(x, y)$ taken at a constant value, say $z = c$.
- A function has many level curves, as one obtains a different level curve for each value of c in the range of $f(x, y)$.
- We can plot the level curves for a bunch of different constants c together in a level curve plot, which is sometimes called a contour plot.

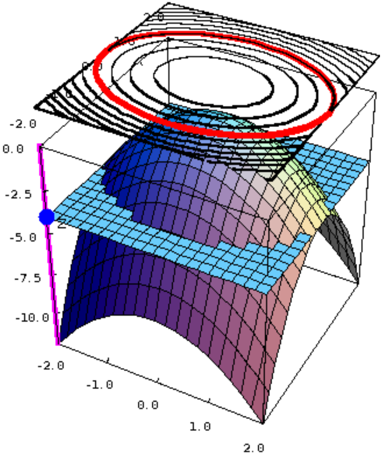
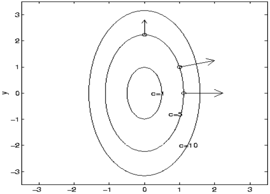


Figure: The graph of the function $f(x, y) = -x^2 - 2y^2$ is shown along with a level curve plot.

- Consider $z = f(x, y) = 4x^2 + y^2$.
- The figure below shows the level curves, defined by $f(x, y) = c$, of the surface.
- The level curves are the ellipses $4x^2 + y^2 = c$.
- As the plot shows, the gradient vector at (x, y) is normal to the level curve through (x, y).



- Now consider a scalar field f differentiable on an open set $\mathbf{S}$ in $\mathbb{R}^3$, and examine one of its level surfaces, $L(c)$.
- Let $\mathbf{a}$ be a point on this surface, and consider a curve Γ which lies on the surface and passes through $\mathbf{a}$.
- We shall prove that the gradient vector $\nabla f(\mathbf{a})$ is normal to this curve at $\mathbf{a}$.

- That is, we shall prove that $\nabla f(\mathbf{a})$ is perpendicular to the tangent vector of Γ at $\mathbf{a}$.

- For this purpose we assume that Γ is described parametrically by a differentiable vector-valued function $\mathbf{r}$ defined on some interval $\mathbb{j}$ in $\mathbb{R}^1$.

- Since Γ lies on the level surface $L(c)$, the function $\mathbf{r}$ satisfies the equation

$$f[\mathbf{r}(t)] = c \text{ for all } t \text{ in } \mathbb{j}.$$

- If $g(t) = f[\mathbf{r}(t)]$ for t in $\mathbb{j}$, the chain rule states that

$$g'(t) = \nabla f[\mathbf{r}(t)].\mathbf{r}'(t).$$

- Since g is a constant on $\mathbb{j}$, we have $g'(t) = 0$ on $\mathbb{j}$. In particular, choosing t_1 so that $\mathbf{r}(t_1) = \mathbf{a}$, we find that

$$\nabla f(\mathbf{a}).\mathbf{r}'(t_1) = 0.$$

- In other words, the gradient of f at $\mathbf{a}$ is perpendicular to the tangent vector $\mathbf{r}'(t_1)$, as asserted.

- Now we take family of curves on the level surface $L(c)$, all passing through the point $\mathbf{a}$.
- According to the foregoing discussion, the tangent vectors of all these curves are perpendicular to the gradient vector $\nabla f(\mathbf{a})$.
- If $\nabla f(\mathbf{a})$ is not the zero vector, these tangent vectors determine a plane, and the gradient $\nabla f(\mathbf{a})$ is normal to this plane.
- This particular plane is called as the tangent plane of the surface $L(c)$ at $\mathbf{a}$.

- We know that a plane through $\mathbf{a}$ with normal vector N consists of all points $\mathbf{x} \in \mathbb{R}^3$ satisfying $N.(\mathbf{x} - \mathbf{a}) = 0$.
- Therefore the tangent plane to the level surface $L(c)$ at $\mathbf{a}$ consists of all $\mathbf{x}$ in $\mathbb{R}^3$ satisfying

$$\nabla f(\mathbf{a}).(\mathbf{x} - \mathbf{a}) = 0.$$

- To obtain a Cartesian equation for this plane we express $\mathbf{x}$, $\mathbf{a}$, and $\nabla f(\mathbf{a})$ in terms of thier components.
- Writing $\mathbf{x} = (x, y, z)$, $\mathbf{a} = (x_1, y_1, z_1)$ and

$$\nabla f(\mathbf{a}) = D_1 f(\mathbf{a})\mathbf{i} + D_2 f(\mathbf{a})\mathbf{j} + D_3 f(\mathbf{a})\mathbf{k},$$

we obtain the Cartesian equation

$$D_1 f(\mathbf{a})(x - x_1) + D_2 f(\mathbf{a})(y - y_1) + D_3 f(\mathbf{a})(z - z_1) = 0.$$

- A similar discussion applies to a scalar fields defined in $\mathbb{R}^2$.
- In Example 3 we proved that the gradient vector $\nabla f(\mathbf{a})$ at a point $\mathbf{a}$ of a level curve is perpendicular to the tangent vector of the curve at $\mathbf{a}$.
- Therefore the tangent line of the level curve $L(c)$ at a point $\mathbf{a} = (x_1, y_1)$ has the Cartesian equation

$$D_1 f(\mathbf{a})(x - x_1) + D_2 f(\mathbf{a})(y - y_1) = 0.$$

The equation of the tangent plane

Consider the surface $z = f(x, y)$. If $Z = f(X, Y)$, then $(X, Y, Z)^T$ is a point on the surface $z = f(x, y)$. If the surface admits a non vertical tangent plane at $(X, Y, Z)^T$, then we say that f is differentiable at $(X, Y)^T$.

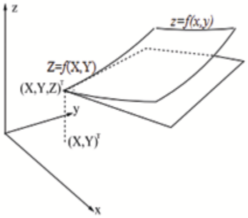


Figure: The tangent plane

If f is differentiable at $(X, Y)^T$ its tangent plane must have equation

$$z - Z = f_x(X, Y)(x - X) + f_y(X, Y)(y - Y).$$

We usually write this in the less precise form

$$z - Z = \frac{\partial f}{\partial x}(X, Y)(x - X) + \frac{\partial f}{\partial y}(X, Y)(y - Y).$$

N.B Partial derivatives are to be evaluated at the point $(X, Y)^T$.

Example

Let $f(x, y) = \dfrac{x - y}{x + y}$.

(a) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Find the equation of the tangent plane to the surface $z = f(x, y)$ where $x = 1$ and $y = 1$.

Example
Solution

(a)

$$
\begin{aligned}
f(x,y) &= \frac{x-y}{x+y} \\
\frac{\partial f}{\partial x} &= \frac{(x+y).1-(x-y).1}{(x+y)^2} \\
&= \frac{2y}{(x+y)^2}. \\
\frac{\partial f}{\partial y} &= \frac{(x+y).(-1)-(x-y).1}{(x+y)^2} \\
&= \frac{-2x}{(x+y)^2}.
\end{aligned}
$$

(b) The tangent plane must have equation

$$z - Z = f_x(X, Y)(x - X) + f_y(X, Y)(y - Y).$$

The equation of the tangent plane to the surface $z = f(x, y)$, where $X = 1$ and $Y = 1$ is

$$z - Z = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1),$$

where $Z = f(1, 1)$. The required equation is

$$z = \frac{1}{2}(x - 1) - \frac{1}{2}(y - 1).$$

Thank you!