Real Analysis III

(MAT312β)

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About course unit

- Course unit: Real Analysis-III(MAT312β)
- Credit value: 2.5
- Number of lecture hours: 30
- Number of tutorial hours: 15
- No prerequisite course unit is required
- Method of assessment: End of semester examination
- Attendance: Both tutorial and lecture will be considered

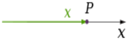
References

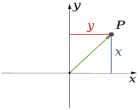
- Applied Calculus by Laurence D. Hoffmann, Gerald L. Bradley, Kenneth H. Rosen. (515 HOF).
- Calculus of several variables by Mclachlan. (515 MCL).
- Mathematical analysis by Apostol, Tom M. (515APO).
- http://www.math.ruh.ac.lk/∼pubudu/

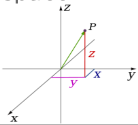
Introduction to n-dimensional space

What is dimension?

- In mathematics, the dimension of a space is informally defined as the minimum number of co-ordinates needed to specify any point within it.
- Thus a line has a dimension of one because only one co-ordinate is needed to specify a point on it.
- A plane has a dimension of two because two co-ordinates are needed to specify a point on it.
- The inside of a sphere is three-dimensional because three co-ordinates are needed to locate a point within this space.







Why do we need higher dimension?

- High-dimensional spaces occur in mathematics and the sciences for many reasons.
- For instance, if you are studying a chemical reaction involving 6 chemicals, you will probably want to store and manipulate their concentrations as a 6-tuple.
- The laws governing chemical reaction rates also demand we do calculus in this 6-dimensional space.

n-dimensional space

- We shall denote by $\mathbb{R}$ the field of real numbers.
- Then we shall use the Cartesian product $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \mathbb{R}$ of ordered n-tuples of real numbers (n factors).
- $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{x} = (x_1, x_2, \ldots, x_n)$.
- Here $\mathbf{x}$ is called a point or a vector, and $x_1, x_2, \ldots, x_n$ are called the coordinates of $\mathbf{x}$.
- The natural number n is called the dimension of the space.

n-dimensional space
Cont...

- $\mathbb{R}^1 \Rightarrow \mathbf{x} = (x_1)$
- $\mathbb{R}^2 \Rightarrow \mathbf{x} = (x_1, x_2)$
- $\mathbb{R}^3 \Rightarrow \mathbf{x} = (x_1, x_2, x_3)$
- $\mathbb{R}^4 \Rightarrow \mathbf{x} = (x_1, x_2, x_3, x_4)$
- $\mathbb{R}^m \Rightarrow \mathbf{x} = (x_1, x_2, ..., x_m)$
- $\mathbb{R}^n \Rightarrow \mathbf{x} = (x_1, x_2, ..., x_n)$

More on n-dimensional space

Let $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ be points in $\mathbb{R}^n$ and let a be a real number. Then we define

1. $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$.
2. $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, ..., x_n - y_n)$.
3. $a\mathbf{x} = (ax_1, ax_2 ..., ax_n)$.

If $\mathbf{x} = (2, -3, 1)$ and $\mathbf{y} = (-4, 1, -2)$ are two points in $\mathbb{R}^3$, then find

(i) $\mathbf{x} + \mathbf{y}$.

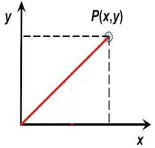
(ii) $\mathbf{x} - \mathbf{y}$.

(iii) $\mathbf{y} + \mathbf{x}$.

(iv) $2\mathbf{x} + 3\mathbf{y}$.

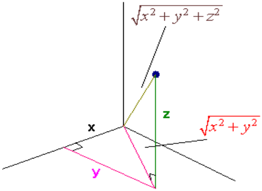
The length of a vector in two dimensional space

- We require some method to measure the magnitude of a vector.
- Based on Pythagorean Theorem, the vector from the origin to the point (4, 5) in two dimensional space has length of $\sqrt{4^2 + 5^2} = \sqrt{41}$.
- The vector from the origin to the point (x, y) has the length $\sqrt{x^2 + y^2}$.
- The length of a vector with two elements is the square root of the sum of each element squared.



The length of a vector in three dimensional space

- The vector from the origin to the point (x, y, z) has the length $\sqrt{x^2 + y^2 + z^2}$.
- The length of a vector with three elements is the square root of the sum of each element squared.



The length of a vector in n-dimensional space

- In $\mathbb{R}^n$, the intuitive notion of length of the vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ is captured by the formula,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

- The magnitude of a vector is sometimes called the length of a vector, or **norm** of a vector.
- Basically, norm of a vector is a measure of distance, symbolized by $\|\mathbf{x}\|$.

Find the distances from the origin to the following vectors.

1. $\mathbf{x} = (2, 4, -1, 1) \in \mathbb{R}^4$
2. $\mathbf{y} = (1, 3, -2, 1, 4) \in \mathbb{R}^5$

The distance between two points in n-dimensional space

- In particular if we let $\|\mathbf{x}\|$ denote the distance from $\mathbf{x} = (x_1, x_2, ..., x_n)$ to the origin $\mathbf{0} = (0, 0, .., 0)$ in $\mathbb{R}^n$, then

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \\ \|\mathbf{x} - \mathbf{0}\| &= \sqrt{(x_1 - 0)^2 + (x_2 - 0)^2 + ... + (x_n - 0)^2}. \end{aligned}$$

- With this notation, the distance from $\mathbf{y} = (y_1, y_2, ..., y_n)$ to $\mathbf{x} = (x_1, x_2, ..., x_n)$ is

$$\|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}.$$

Let $\mathbf{x} = (1, 2, -3)$ and $\mathbf{y} = (3, -2, 1)$. Then find the distance from

(i) $\mathbf{x}$ to the origin.

(ii) $\mathbf{x}$ to $\mathbf{y}$.

Norm of a scalar times a vector

Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. If α is a scalar, how does the norm of $\alpha\mathbf{x}$ compare to the norm of $\mathbf{x}$?

Norm of a scalar times a vector
Proof

If $\mathbf{x} = (x_1, x_2, ..., x_n)$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$.

$$\begin{aligned} \|\alpha\mathbf{x}\| &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + ... + (\alpha x_n)^2} \\ &= \sqrt{\alpha^2[(x_1)^2 + (x_2)^2 + ... + (x_n)^2]} \\ &= \sqrt{\alpha^2}.\sqrt{[(x_1)^2 + (x_2)^2 + ... + (x_n)^2]} \\ &= |\alpha|.\|\mathbf{x}\| \end{aligned}$$

Thus, multiplying a vector by a scalar α multiplies its norm by $|\alpha|$.

Unit vector

- Any vector whose length is 1 is called a unit vector.
- Let $\mathbf{x}$ be a given nonzero vector and consider the scalar multiple $\frac{1}{\|\mathbf{x}\|}\mathbf{x}$.
- Applying the above result (with $\alpha = \frac{1}{\|\mathbf{x}\|}$), the norm of the vector $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is

$$\begin{aligned} \|\frac{1}{\|\mathbf{x}\|}\mathbf{x}\| &= |\frac{1}{\|\mathbf{x}\|}|\,\|\mathbf{x}\| \\ &= \frac{1}{\|\mathbf{x}\|}\|\mathbf{x}\| = 1 \end{aligned}$$

- Thus, for any nonzero vector $\mathbf{x}$, $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is a unit vector.

Unit vector
Example

Find the vector $\mathbf{v}$ in $\mathbb{R}^2$ whose length is 10 and which has the same direction as $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

First, find the unit vector in the same direction as $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, and then multiply this unit vector by 10. The unit vector in the direction of $\mathbf{u}$ is

$$\begin{aligned}\hat{\mathbf{u}} &= \frac{\mathbf{u}}{\|\mathbf{u}\|} \\ &= \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{3^2 + 4^2}} \\ &= \frac{3\mathbf{i} + 4\mathbf{j}}{5} \\ &= \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\end{aligned}$$

Therefore $\mathbf{v} = 10\hat{\mathbf{u}} = 10\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = 6\mathbf{i} + 8\mathbf{j}$.

Unit vector
Past paper 2013

Find the vector $\mathbf{v}$ in $\mathbb{R}^3$ whose magnitude is $\sqrt{2}/\log(\sqrt{5})$ and has the same direction as $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

First, find the unit vector in the direction of $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$, and then multiply this unit vector by $\sqrt{2}/\log(\sqrt{5})$. The unit vector in the direction of $\mathbf{u}$ is

$$\begin{aligned}
\hat{\mathbf{u}} &= \frac{\mathbf{u}}{\|\mathbf{u}\|} \\
&= \frac{-2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{(-2)^2 + 3^2 + 6^2}} \\
&= \frac{-2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7} \\
&= \frac{-2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + +\frac{6}{7}\mathbf{k}
\end{aligned}$$

Therefore

$$\mathbf{v} = (\sqrt{2}/\log(\sqrt{5}))\hat{\mathbf{u}} = (\sqrt{2}/\log(\sqrt{5})).\left(\frac{-2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + +\frac{6}{7}\mathbf{k}\right).$$

Inner product

- An inner product is a generalization of the dot product.
- In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.
- The inner product is usually denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.
- In $\mathbb{R}^n$, where the inner product is given by the dot product,

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle \\
&= x_1 y_1 + x_2 y_2 + \dots + x_n y_n \\
&= \sum_{k=1}^{n} x_k y_k
\end{aligned}
$$

Inner product
Example

What is the inner product of the vectors $\mathbf{x} = (-2, 1, 4, 1)$ and $\mathbf{y} = (1, 3, 2, 4)$ in $\mathbb{R}^4$?

Proposition

For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and scalar α. Then

1. $\mathbf{x}.\mathbf{y} = \mathbf{y}.\mathbf{x}$
2. $\mathbf{x}.(\mathbf{y} + \mathbf{z}) = \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}$
3. $(\alpha\mathbf{x}).\mathbf{y} = \alpha(\mathbf{x}.\mathbf{y})$
4. $\mathbf{0}.\mathbf{x} = 0$
5. $\mathbf{x}.\mathbf{x} \geq 0$
6. $\mathbf{x}.\mathbf{x} = \|\mathbf{x}\|^2$

Proposition

Poof of (1)

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n)$ and $\mathbf{z} = (z_1, z_2, ..., z_n)$.

$$\begin{aligned}
\mathbf{x}.\mathbf{y} &= (x_1, x_2, ..., x_n).(y_1, y_2, ..., y_n) \\
&= x_1 y_1 + x_2 y_2 + ... + x_n y_n \\
&= y_1 x_1 + y_2 x_2 + ... + y_n x_n \\
&= (y_1, y_2, ..., y_n).(x_1, x_2, ..., x_n) \\
&= \mathbf{y}.\mathbf{x}
\end{aligned}$$

Proposition
Poof of (2)

$$
\begin{aligned}
&= \mathbf{x}.(\mathbf{y}+\mathbf{z}) \\
&= (x_1, x_2, ..., x_n).[(y_1, y_2, ..., y_n) + (z_1, z_2, ..., z_n)] \\
&= (x_1, x_2, ..., x_n).(y_1 + z_1, y_2 + z_2, ..., y_n + z_n) \\
&= [x_1(y_1 + z_1) + x_2(y_2 + z_2) + ... + x_n(y_n + z_n)] \\
&= [(x_1 y_1 + x_1 z_1) + (x_2 y_2 + x_2 z_2) + ... + (x_n y_n + x_n z_n)] \\
&= (x_1 y_1 + x_2 y_2 + ... + x_n y_n) + (x_1 z_1 + x_2 z_2 + ... + x_n z_n) \\
&= [(x_1, x_2, ..., x_n).(y_1, y_2, ..., y_n)] + [(x_1, x_2, ..., x_n).(z_1, z_2, ..., z_n)] \\
&= \mathbf{x}.\mathbf{y} + \mathbf{x}.\mathbf{z}
\end{aligned}
$$

Proposition
Poof of (3)

$$
\begin{aligned}
(\alpha \mathbf{x})\mathbf{y} &= [\alpha(x_1, x_2, ..., x_n)].(y_1, y_2, ..., y_n) \\
&= (\alpha x_1, \alpha x_2, ..., \alpha x_n).(y_1, y_2, ..., y_n) \\
&= (\alpha x_1 y_1 + \alpha x_2 y_2 + ... + \alpha x_n y_n) \\
&= \alpha(x_1 y_1 + x_2 y_2 + ... + x_n y_n) \\
&= \alpha(\mathbf{x}.\mathbf{y})
\end{aligned}
$$

Proposition
Poof of (4)

$$
\begin{aligned}
\mathbf{0}.\mathbf{x} &= (0, 0, ..., 0).(x_1, x_2, ..., x_n) \\
&= 0x_1 + 0x_2 + ... + 0x_n \\
&= 0 + 0 + ... + 0 \\
&= 0
\end{aligned}
$$

Proposition

Poof of (5)

$$\begin{aligned}\mathbf{x}.\mathbf{x} &= (x_1, x_2, ..., x_n).(x_1, x_2, ..., x_n) \\ &= x_1^2 + x_2^2 + ... + x_n^2 \geq 0 \\ \text{Therefore } \mathbf{x}.\mathbf{x} &\geq 0\end{aligned}$$

Proposition
Poof of (6)

$$\begin{aligned}
\mathbf{x}.\mathbf{x} &= (x_1, x_2, ..., x_n).(x_1, x_2, ..., x_n) \\
\mathbf{x}.\mathbf{x} &= x_1^2 + x_2^2 + ... + x_n^2 \rightarrow \text{(A)} \\
\|\mathbf{x}\| &= \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \\
\|\mathbf{x}\|^2 &= x_1^2 + x_2^2 + ... + x_n^2 \rightarrow \text{(B)} \\
\text{From (A) and (B)} & \\
\mathbf{x}.\mathbf{x} &= \|\mathbf{x}\|^2
\end{aligned}$$

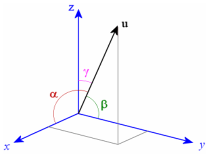
Direction cosine in $\mathbb{R}^3$

The direction cosines (or directional cosines) of a vector are the cosines of the angles between the vector and the three coordinate axes. If **u** is a vector

$$\mathbf{u} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},$$

then

$$\begin{aligned}
\cos\alpha &= \frac{x_1}{\|\mathbf{u}\|} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
\cos\beta &= \frac{x_2}{\|\mathbf{u}\|} = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
\cos\gamma &= \frac{x_3}{\|\mathbf{u}\|} = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}
\end{aligned}$$



Direction cosine in $\mathbb{R}^n$

- In general $\mathbf{x} \in \mathbb{R}^n$ can be considered as either a vector in $\mathbb{R}^n$ or as a point in $\mathbb{R}^n$ starting at the origin with length $\|\mathbf{x}\|$.
- If $\mathbf{x} \neq \mathbf{0}$ then we get

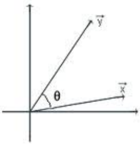
$$\mathbf{c} = \left(\frac{x_1}{\|\mathbf{x}\|}, \frac{x_2}{\|\mathbf{x}\|}, ..., \frac{x_n}{\|\mathbf{x}\|} \right)$$

  the direction of $\mathbf{x}$.
- The co-ordinates of $\mathbf{c}$, that is $\frac{x_k}{\|\mathbf{x}\|}$, $k = 1, 2, ..., n$ are called directional cosines.

The angle between two vectors

- Angle should take any two vectors **x** and **y** and produce a real number, $\theta \in [0, 2\pi)$.
- Angle should not depend on the lengths (norms) of **x** and **y**.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$\cos\theta = \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$



If $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -2, 2)$, find the angle between $\mathbf{x}$ and $\mathbf{y}$.

The angle between a vector and a co-ordinate axis

- Let $\mathbf{x} \in \mathbb{R}^n$, let $\alpha_k, k = 1, 2, 3, ..., n$ be the angle between $\mathbf{x}$ and the k^{th} axis.
- Then α_k is the angle between the standard basis vector $\mathbf{e}_k$ and $\mathbf{x}$.
- Thus we have

$$\cos(\alpha_k) = \frac{\mathbf{x}.\mathbf{e_k}}{\|\mathbf{x}\|\|\mathbf{e_k}\|} = \frac{x_k}{\|\mathbf{x}\|}.$$

The angle between a vector and a co-ordinate axis
Example

Find the angle between $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and the x axis.

Orthogonal vectors

- Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$. Then if $\mathbf{x}.\mathbf{y} = 0$,

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= \frac{0}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= 0 \\
\cos\theta &= \cos\frac{\pi}{2} \\
\theta &= \frac{\pi}{2}
\end{aligned}$$

- The angle between $\mathbf{x}$ and $\mathbf{y}$ is $\dfrac{\pi}{2}$.

Orthogonal vectors
Cont...

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be orthogonal (or perpendicular) if $\mathbf{x}.\mathbf{y} = 0$.

$$\mathbf{x}.\mathbf{y} = 0 \Longrightarrow \mathbf{x} \perp \mathbf{y}$$

Orthogonal vectors
Example

1. Show that $\mathbf{x} = (-1, -2)$ and $\mathbf{y} = (1, 2)$ are both orthogonal to $\mathbf{z} = (2, -1)$ in $\mathbb{R}^2$.
2. Show that $\mathbf{x} = (1, -1, 1, -1)$ and $\mathbf{y} = (1, 1, 1, 1)$ are perpendicular in $\mathbb{R}^4$.

Parallel vectors

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \neq 0$ is a scalar. Then we say that $\mathbf{x}$ and $\mathbf{y}$ are parallel if $\mathbf{x} = \alpha \mathbf{y}$.

$$\mathbf{x} = \alpha \mathbf{y} \Longrightarrow \mathbf{x} \parallel \mathbf{y}$$

Parallel vectors
Example

1. Suppose $\mathbf{x} = (2, 1, 3)$ and $\mathbf{y} = (4, 2, 6)$ in $\mathbb{R}^3$. We can write down $\mathbf{x} = \dfrac{1}{2}\mathbf{y}$. Therefore $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^3$.

2. Suppose $\mathbf{x} = (8, -2, 6, -4)$ and $\mathbf{y} = (24, -6, 18, -12)$ in $\mathbb{R}^4$. We can write down $\mathbf{y} = 3\mathbf{x}$. Therefore $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^4$.

Cauchy-Schwarz inequality

- The Cauchy-Schwarz inequality is a useful inequality encountered in many different situations.
- It is considered to be one of the most important inequalities in all of mathematics.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$|\mathbf{x}.\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\| \tag{1}$$

is called the Cauchy-Schwarz inequality.

When $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, (1) holds with equality.

Let us assume that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are fixed vectors with $\mathbf{y} \neq \mathbf{0}$ (or $\mathbf{x} \neq \mathbf{0}$).

Take a real number $t \in \mathbb{R}$ and define the function

$$\begin{aligned} f(t) &= (\mathbf{x} + t\mathbf{y}).(\mathbf{x} + t\mathbf{y}) \\ &= (\mathbf{x} + t\mathbf{y})^2 \text{ (Therefore } f(t) \geq 0) \\ f(t) &= \mathbf{x}.\mathbf{x} + \mathbf{x}.t\mathbf{y} + t\mathbf{y}.\mathbf{x} + .t\mathbf{y}.t\mathbf{y} \\ f(t) &= \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y})t + t^2\|\mathbf{y}\|^2 \end{aligned}$$

Hence $f(t)$ is a quadratic function of t with at most one root.

The roots of $f(t)$ are given by

$$\frac{-2(\mathbf{x}.\mathbf{y}) \pm \sqrt{4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2}}{2\|\mathbf{y}\|^2}.$$

Since $f(t) \geq 0 \Longrightarrow 4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$.

$$\begin{aligned} 4(\mathbf{x}.\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 &\leq 0 \\ (\mathbf{x}.\mathbf{y})^2 &\leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \\ |\mathbf{x}.\mathbf{y}| &\leq \|\mathbf{x}\|\|\mathbf{y}\| \end{aligned}$$

$$
\begin{aligned}
|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| \quad &\iff \quad f(t) = 0 \text{ for some value of } t \\
&\iff \quad f(t) = 0 \\
&\iff \quad (\mathbf{x} + t\mathbf{y})^2 = 0 \\
&\iff \quad (\mathbf{x} + t\mathbf{y}) = 0 \\
&\iff \quad \mathbf{x} = -t\mathbf{y}
\end{aligned}
$$

Hence the inequality (1) becomes an equality iff either **x** is a scalar multiple of **y** or **y** is a scalar multiple of **x**.

$$|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\| \quad \iff \quad \mathbf{x} \parallel \mathbf{y}$$

$|\mathbf{x}.\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$ if $\mathbf{x}$ and $\mathbf{y}$ are parallel.

$$
\begin{aligned}
|\mathbf{x}.\mathbf{y}| &= \|\mathbf{x}\|\|\mathbf{y}\| \\
\frac{|\mathbf{x}.\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} &= \frac{\|\mathbf{x}\|\|\mathbf{y}\|}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\frac{|\mathbf{x}.\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|} &= 1 \\
\frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} &= \pm 1 \longrightarrow (\mathrm{A}) \\
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \longrightarrow (\mathrm{B})
\end{aligned}
$$

From (A) and (B)

$$\begin{aligned} \cos\theta &= \pm 1 \\ \cos\theta &= 1 & \cos\theta = -1 \\ \cos\theta &= \cos 0 & \cos\theta = \cos\pi \\ \theta &= 0 & \theta = \pi \end{aligned}$$

Show that $\mathbf{x} = (1, -3)$ and $\mathbf{y} = (-2, 6)$ are parallel in $\mathbb{R}^2$.

Cauchy-Schwarz inequality

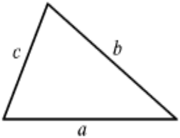
Remark 2⇒Example⇒Solution

$$
\begin{aligned}
\cos\theta &= \frac{\mathbf{x}.\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \\
\cos\theta &= \frac{(1,-3).(-2,6)}{\sqrt{1^2+(-3)^2}\sqrt{(-2)^2+6^2}} \\
\cos\theta &= \frac{-20}{2\times 10} \\
\cos\theta &= -1 \\
\cos\theta &= \cos\pi \\
\theta &= \pi \implies \mathbf{x} \parallel \mathbf{y}
\end{aligned}
$$

Triangle inequality
A property for any triangle

- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- In the figure, the following inequalities hold.

  1. $a + b > c$
  2. $a + c > b$
  3. $b + c > a$



Triangle inequality

Motivative example 1

Check whether it is possible to have a triangle with the given side lengths 4, 5, 7.

Triangle inequality

Motivative example 1⇒Solution

We should add any two sides and see if it is greater than the other side.

- The sum of 4 and 5 is 9 and 9 is greater than 7.
- The sum of 4 and 7 is 11 and 11 is greater than 5.
- The sum of 5 and 7 is 12 and 12 is greater than 4.
- These sides 4, 5, 7 satisfy the above property.
- Therefore, it is possible to have a triangle with sides 4, 5, 7.

Check whether the given side lengths form a triangle 2, 5, 9.

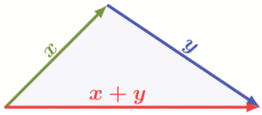
Check whether the sides satisfy the above property.

- The sum of 2 and 5 is 7 and 7 is less than 9.
- This set of side lengths does not satisfy the above property.
- Therefore, these lengths do not form a triangle.

Triangle inequality
For real numbers

- **The triangle inequality states that for any triangle, the sum of the lengths of any two sides must be greater than the length of the remaining side**.
- The triangle inequality requires that the absolute value satisfy for any real numbers x and y:

$$|x + y| \leq |x| + |y|.$$



Triangle inequality
For norms of vectors

- The triangle inequality is a defining property of norms of vectors.
- That is, the norm of the sum of two vectors is at most as large as the sum of the norms of the two vectors.
- For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) \text{ (Since } \|\mathbf{x}\|^2 = \mathbf{x}.\mathbf{x}) \\
&= \mathbf{x}.\mathbf{x}+\mathbf{x}.\mathbf{y}+\mathbf{y}.\mathbf{x}+\mathbf{y}.\mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y}) + \|\mathbf{y}\|^2 \rightarrow \text{(A)} \\
|\mathbf{x}.\mathbf{y}| &\leq \|\mathbf{x}\|\,\|\mathbf{y}\| \text{ (CS inequality)} \\
\mathbf{x}.\mathbf{y} &\leq \|\mathbf{x}\|\,\|\mathbf{y}\| \rightarrow \text{(B)} \\
\text{From (A) and (B), we have} & \\
\|\mathbf{x}+\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + 2(\|\mathbf{x}\|\,\|\mathbf{y}\|) + \|\mathbf{y}\|^2 \\
\|\mathbf{x}+\mathbf{y}\|^2 &\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \\
\|\mathbf{x}+\mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}
$$

From (A)

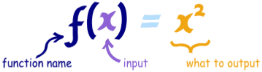
$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(\mathbf{x}.\mathbf{y}) + \|\mathbf{y}\|^2,$$

iff $\mathbf{x}.\mathbf{y} = 0$, that is iff $\mathbf{x}\perp\mathbf{y}$, then

$$\begin{aligned}
&\Rightarrow \quad \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2(0) + \|\mathbf{y}\|^2 \\
&\Rightarrow \quad \|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \\
&\Rightarrow \quad \text{Pythagorean theorem}
\end{aligned}$$

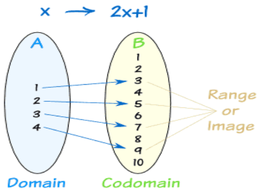
What is a function?

- In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.
- What can go into a function is called the **domain**.
- What may possibly come out of a function is called the **codomain**.
- What actually comes out of a function is called the **range**.



What is a function?
Example

- The set "A" is the Domain $\Rightarrow \{1, 2, 3, 4\}$.
- The set "B" is the Codomain $\Rightarrow \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
- The actual values produced by the function is the Range $\Rightarrow \{3, 5, 7, 9\}$.



Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

- We shall consider a function $\mathbf{f}$ with domain in n-space $\mathbb{R}^n$ and with range in m-space $\mathbb{R}^m$.
- It can be denoted as $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$.
- Both n and m are natural numbers and they can have different values.

- Then $f : \mathbb{R} \to \mathbb{R}$.
- Such a function is called as real-valued function of a real variable.
- In other words, it is a function that assigns a real number to each member of its domain.
- **Eg:** $f(x) = 2x + 1, f(x) = x^2 + 5, f(u) = 5u - 8$

- Then $\mathbf{f} : \mathbb{R} \to \mathbb{R}^m$.
- It is called as vector-valued function of a real variable.
- A common example of a vector valued function is one that depends on a single real number parameter t, often representing time, producing a vector $\mathbf{v}(t)$ as the result.
- **Eg:** $\mathbf{f}(t) = h(t)\mathbf{i} + g(t)\mathbf{j}$, where $h(t)$ and $g(t)$ are the coordinate functions of the parameter t.

- Then $f : \mathbb{R}^n \to \mathbb{R}$.
- The function is called as a real-valued function of a vector variable or, more briefly a **scalar field**.
- **Eg:** If $f : \mathbb{R}^3 \to \mathbb{R}$ the level surface of value c is the set of points $\{(x, y, z) : f(x, y, z) = c\}$.
- **Eg:** The temperature distribution throughout space, the pressure distribution in a fluid.

- Then $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$.
- The function is called as a vector-valued function of a vector variable or, more briefly a **vector field**.
- **Eg:** A function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ can be defined by

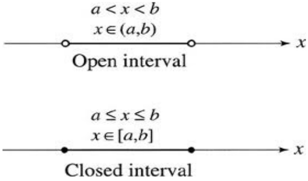
$$\mathbf{f}(x_1, x_2, x_3) = \left(\cos\sqrt{x_1^2 + x_2^2 + x_3^3 - 1}, \sin\sqrt{x_1^2 + x_2^2 + x_3^3 - 1}\right).$$

- **Eg:** Velocity field of a moving fluid, Magnetic fields, A gravitational field generated by any massive object.

- Scalars are denoted by light-faced characters.
- Vectors are denoted by bold-faced characters.
- If f is a scalar field defined at a point $\mathbf{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$, the notations $f(\mathbf{x})$ and $f(x_1, ..., x_n)$ are both used to denote the value of f at that particular point.
- If $\mathbf{f}$ is a vector field defined at a point $\mathbf{x} = (x_1, ..., x_n)$ in $\mathbb{R}^n$, the notations $\mathbf{f}(\mathbf{x})$ and $\mathbf{f}(x_1, ..., x_n)$ are both used to denote the value of $\mathbf{f}$ at that particular point.

Open and closed intervals in $\mathbb{R}$

- Given two real numbers a and b with $a < b$, the closed interval $[a, b]$ is defined as the set of all real numbers x such that $a \leq x$ and $x \leq b$, or more concisely, $a \leq x \leq b$.
- The open interval (a, b) is defined as the set of all real numbers x such that $a < x < b$.
- The difference between the closed interval $[a, b]$ and the open interval (a, b) is that the end points a and b are elements of $[a, b]$ but are not elements of (a, b).



The importance of open intervals

- Open intervals play an important role in calculus of a function of a single real variable.
- Recall the definition of a local minimum (or maximum).
- Let $f : \mathbb{D} \to \mathbb{R}$, $(\mathbb{D} \subset \mathbb{R})$. Then we say that a point $c \in \mathbb{D}$ is a local minimum of f if there is an **open interval** U such that $c \in U$ and $f(x) \geq f(c)\ \forall x \in U \cap D$.
- Open intervals are used in many other definitions.

Generalization of open intervals into $\mathbb{R}^n$

- The concept of the open interval in $\mathbb{R}$ can be generalized to a subset of $\mathbb{R}^n$.
- The generalized version of the open interval is called as **open sets**.
- A number of important results that we shall obtain on the functions of several variables (scalar or vector field) are only true when the domains of these functions are open sets.

Open balls

- Let $\mathbf{a} = (a_1, a_2, ..., a_n)$ be a given point in $\mathbb{R}^n$ and let r be a given positive number. The set of all points $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ such that,

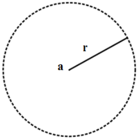
$$\|\mathbf{x} - \mathbf{a}\| < r,$$

  is called an open n-ball of radius r and center $\mathbf{a}$.

- We denote this set by $\mathbf{B}(\mathbf{a})$ or by $\mathbf{B}(\mathbf{a}; r)$.

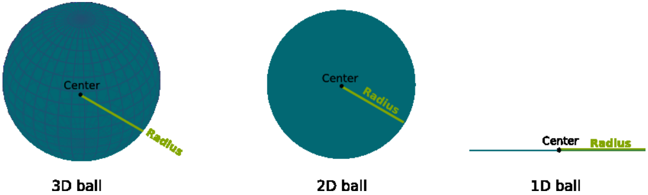
- The ball $\mathbf{B}(\mathbf{a}; r)$ consists of all points whose distance from $\mathbf{a}$ is less than r. So it can be written in symbolic form as,

$$\mathbf{B}(\mathbf{a}; r) = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x} - \mathbf{a}\| < r\}.$$



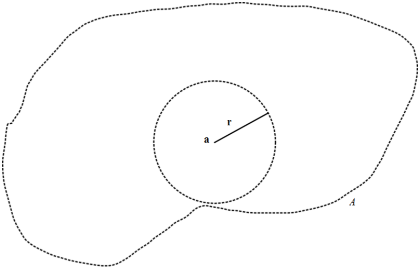
Open balls
Example

- In $\mathbb{R}^1$
  Open ball $\Rightarrow$ an open interval with center at a.
- In $\mathbb{R}^2$
  Open ball $\Rightarrow$ a circular disk with center at $\mathbf{a}$ and radius r.
- In $\mathbb{R}^3$
  Open ball $\Rightarrow$ a spherical solid with center at $\mathbf{a}$ and radius r.



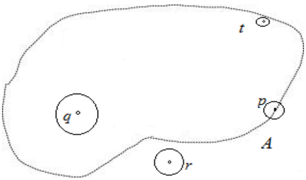
An interior point

Let A be a subset of $\mathbb{R}^n$, and assume that $\mathbf{a} \in A$. Then we say that $\mathbf{a}$ is an interior point of A if there is an open n-ball with center at $\mathbf{a}$, all of whose points belong to A.



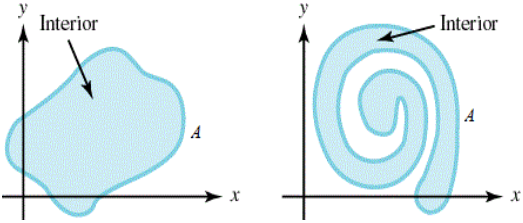
An interior point
Example

- Point t is an interior point.
- Point q is an interior point.
- Point r is not an interior point.
- Point p is not an interior point.



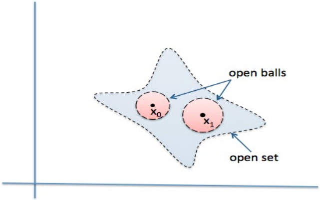
The interior of a set

Let A be a subset of $\mathbb{R}^n$. The set of all interior points of A is called the interior of A and it is denoted by *int* A or A^0.



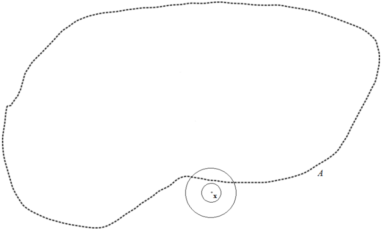
An open set

A set A in $\mathbb{R}^n$ is called open if all its points are interior points. In other words, A is open if and only if $A = int\ A$.



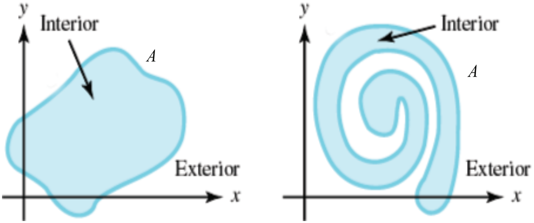
An exterior point

Let A be a subset of $\mathbb{R}^n$. A point $\mathbf{x}$ is said to be exterior to a set A in $\mathbb{R}^n$ if there is an n-ball $\mathbf{B}(\mathbf{x})$ containing no points of A.



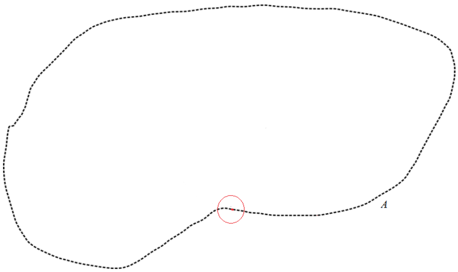
The exterior of a set

Let A be a subset of $\mathbb{R}^n$. The set of all points in $\mathbb{R}^n$ exterior to A is called the exterior of A and it is denoted by *ext* A.



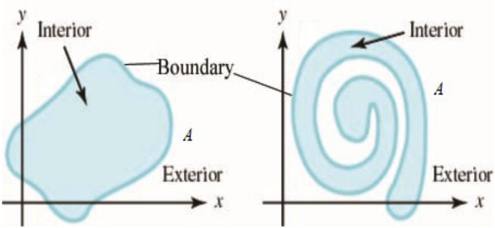
A boundary point

Let A be a subset of $\mathbb{R}^n$. A point which is neither exterior to A nor an interior point of A is called a boundary point of A.

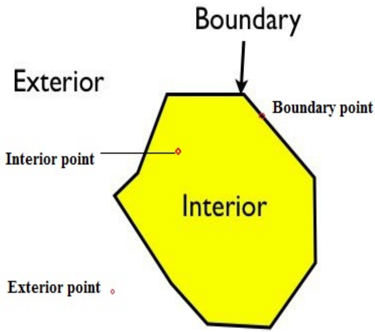


The boundary

Let A be a subset of $\mathbb{R}^n$. The set of all boundary points of A is called the boundary of A and it is denoted by ∂A.



Summary



Boundary
Exterior
Boundary point
Interior point
Interior
Exterior point

How to show a given set is an open set?

1. Any $\mathbf{x} \in \mathbb{S}$ is an interior point.
2. Any $\mathbf{x} \in \mathbb{S}$ is neither a boundary nor an exterior point.
3. $\mathbb{S}$ is open $\Leftrightarrow \mathbb{S}^c$ is closed.

Example

Let A_1 and A_2 are subset of $\mathbb{R}$ and both are open. Then show that the Cartesian product $A_1 \times A_2$ in $\mathbb{R}^2$ defined by,

$$A_1 \times A_2 = \{(a_1, a_2) | a_1 \in A_1 \text{ and } a_2 \in A_2\},$$

is also open.

Example
Solution

- To prove this, choose any point $\mathbf{a} = (a_1, a_2)$ in $A_1 \times A_2$.
- We must show that $\mathbf{a}$ is an interior point of $A_1 \times A_2$.
- A_1 is open in $\mathbb{R} \Rightarrow$ There is a 1-ball $\mathbf{B}(a_1; r_1)$ in A_1.
- A_2 is open in $\mathbb{R} \Rightarrow$ There is a 1-ball $\mathbf{B}(a_2; r_2)$ in A_2.
- Let $r = min\{r_1, r_2\}$.
- We can easily show that the 2-ball $\mathbf{B}(\mathbf{a}; r) \subseteq A_1 \times A_2$.

Example

Solution$\Rightarrow$Cont...

- In fact, if $\mathbf{x} = (x_1, x_2)$ is any point of $\mathbf{B}(\mathbf{a}; r)$ then $\|\mathbf{x} - \mathbf{a}\| < r$.
- So $|x_1 - a_1| < r_1 \Rightarrow \; x_1 \in \mathbf{B}(a_1; r_1)$.
- And $|x_2 - a_2| < r_2 \Rightarrow \; x_2 \in \mathbf{B}(a_2; r_2)$.
- Therefore $x_1 \in A_1$ and $x_2 \in A_2$, so $(x_1, x_2) \in A_1 \times A_2$.
- This proves that every point of $\mathbf{B}(\mathbf{a}; r)$ is in $A_1 \times A_2$.
- Therefore every point of $A_1 \times A_2$ is an interior point.
- So, $A_1 \times A_2$ is open.

Thank you !