Mathematical Computing

IMT2b2β

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Differential Equations

Types of Differential Equations

Differential equations can basically be classified as

- **ordinary differential equation** and
- **partial differential equation**.

- A differential equation that contains only one independent variable is considered an **ordinary differential equation**.
- **Eg:** $\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2 \quad y' = x^2 - 1 \quad \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = x + 1$
- These types of differential equations can be either **linear** or **nonlinear**.
- An ordinary differential equation is **linear** if it is written in the following general form:

$$a_n(x)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}} + ... + a_1(x)y + a_0(x) = f(x).$$

- A differential equation that contains only two or more independent variables is considered a **partial differential equation**.
- **Eg:** $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 1$ $\qquad \dfrac{\partial R}{\partial u} = 3w^2 \dfrac{\partial R}{\partial w} - 5v \dfrac{\partial R}{\partial v}.$

The order of any differential equation is the highest order of the derivative present in the equation.

- $\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = x + 1 \Rightarrow$ ODE of oder 2.
- $y''' + 3xy'' + \frac{6}{x}y = 1 - x^3 \Rightarrow$ODE of oder 3.
- $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 3yx = \frac{\sin x}{x} \Rightarrow$ODE of oder 1.
- $\frac{\partial R}{\partial u} = 3w^2 \frac{\partial R}{\partial w} - 5v \frac{\partial R}{\partial v} \Rightarrow$ PDE of oder 1.

Ordinary Differential Equations

- An ordinary differential equation is an equation where the unknown is not a number but a function.
- First-order ODEs contain first derivatives and can be written as $y' = f(x, y)$.
- Second-order ODEs contain first and second derivatives and can be written as $y'' = f(x, y, y')$.

There are two ways to represent ordinary differential equations in Maxima.

- The first way is to represent the derivatives by:
  'diff(y,x,1)$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ **'diff(y,x,2)**$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
- The second way is to write $y(x)$ explicitly as a function of x:
  diff(y(x),x,1)$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ **diff(y(x),x,2)**$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

- Routine **ode2** can be used to solve first-order ODEs.
- $\mathbf{y}' = \mathbf{f}(\mathbf{x}, \mathbf{y}) \Rightarrow$ **ode2(eqn,y,x)**.
- It takes three arguments: an ODE given by **eqn**, the dependent variable **y**, and the independent variable **x**.
- The function tries various integration methods to find an analytic solution.

- When successful, it returns either an explicit or implicit solution for the dependent variable.
- %**c** is used to represent the integration constant of first-order equations.
- If routine **ode2** cannot obtain a solution for whatever reason, it returns **false**.
- Routine **ode2** stores information about the method used to solve the differential equations in variable **method**.

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x + 2$

(ii) $y' = x^2 - 1$

(iii) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3yx = \dfrac{\sin x}{x}$

(iv) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(k - y)$

(v) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 11xy = x^3y^3$

(vi) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y$

- Routine **ode2** can also be used to solve second-order differential equations.
- $\mathbf{y''} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{y'}) \Rightarrow$ **ode2(eqn,y,x)**.
- It takes three arguments: an ODE given by **eqn**, the dependent variable **y**, and the independent variable **x**.
- When successful, it returns either an explicit or implicit solution for the dependent variable.
- In this case the two integration constants are represented by **%k1** and **%k2**.

(i) $y'' + y' - 6y = 0$

(ii) $3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$

(iii) $4y'' + 12y' + 9y = 0$

(iv) $y'' - 6y' + 13y = 0$

Initial Value Problems of First-Order

- An initial-value problem for the first-order consists of finding a solution y of the differential equation that also satisfies initial condition of the form:

$$y(x_0) = y_0.$$

- Where y_0 is given constant.

- Initial value problems of first order can be solved by routine **ic1**.

$$\mathbf{y}' = \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad \Rightarrow \quad \mathbf{sol} : \mathbf{ode2}(\mathbf{eqn}, \mathbf{y}, \mathbf{x})$$
$$\mathbf{sol}, \mathbf{y}(\mathbf{x_0}) = \mathbf{y_0} \quad \Rightarrow \quad \mathbf{ic1}(\mathbf{sol}, \mathbf{x} = \mathbf{x_0}, \mathbf{y} = \mathbf{y_0})$$

- It takes a general solution to the equation as found by **ode2** as its first argument.
- The second argument $\mathbf{x_0}$ gives an initial value for the independent in the form $\mathbf{x} = \mathbf{x_0}$ and the third argument $\mathbf{y_0}$ gives the initial value for the dependent variable in the form $\mathbf{y} = \mathbf{y_0}$.

Find the solution of the initial value problem:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}+3yx=\frac{\sin x}{x}, \qquad y(\pi)=0.$$

Initial Value Problems of Second-Order

- An initial-value problem for the second-order consists of finding a solution y of the differential equation that also satisfies initial conditions of the form:

$$y(x_0) = y_0 \qquad y'(x_0) = y_1.$$

- Where y_0 and y_1 are given constants.

- Initial value problems of second order can be solved by routine **ic2**.

$$
\begin{aligned}
\mathbf{y}'' = \mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{y}') \quad &\Rightarrow \quad \mathbf{sol} : \mathbf{ode2}(\mathbf{eqn},\mathbf{y},\mathbf{x}) \\
\mathbf{sol} \quad & \\
\mathbf{y}(\mathbf{x_0}) = \mathbf{y_0} \quad &\Rightarrow \quad \mathbf{ic2}(\mathbf{sol},\mathbf{x}=\mathbf{x_0},\mathbf{y}=\mathbf{y_0},'\mathbf{diff}(\mathbf{y},\mathbf{x}) = \mathbf{y_1}) \\
\mathbf{y}'(\mathbf{x_0}) = \mathbf{y_1} \quad &
\end{aligned}
$$

- **ic2** has an additional forth argument $\mathbf{y_1}$ that gives the initial value of the first derivative of the dependent variable with respect to the independent variable in the form **'diff(y,x)**$= \mathbf{y_1}$.

(i) Solve the initial-value problem:

$$y'' + y(y')^3 = 0 \quad y(0) = 0 \quad y'(0) = 2.$$

(ii) Solve the initial-value problem:

$$y'' + y' - 6y = 0 \quad y(0) = 1 \quad y'(0) = 0.$$

Linear System of ODEs

desolve(**eqn**, **y**)
desolve([**eqn_1**, ..., **eqn_n**], [**y_1**, .., **y_n**])

- The function **desolve**, solves systems of linear ordinary differential equations using Laplace transform.
- Here the **eqn**'s are differential equations in the dependent variables **y_1**, ..., **y_n**.
- The functional dependence of **y_1**, ..., **y_n** on an independent variable, for instance **x**, must be explicitly indicated in the variables and its derivatives.

Solve the following system of linear ODEs.

$$x'(t) = x(t) - 2y(t)$$
$$y'(t) = -x(t) + y(t)$$

Linear System of ODEs

Example⇒Correct Syntax for **desolve**

- Following would not be the correct way to define two equations:

$$eqn1 \quad : \quad 'diff(x,t) = x - 2 * y;$$
$$eqn2 \quad : \quad 'diff(y,t) = -x + y;$$

- The correct way would be:

$$eqn1 \quad : \quad diff(x(t),t) = x(t) - 2 * y(t);$$
$$eqn2 \quad : \quad diff(y(t),t) = -x(t) + y(t);$$

- The call to the function **desolve** would then be:

$$desolve([eqn1, eqn2], [x(t), y(t)]);$$

Linear System of ODEs with Initial Conditions

- If initial conditions are known, they can be supplied by using **atvalue**.
- Notice that the conditions must be given before the equations are solved.

Solve following systems of linear ODEs with given initial conditions.

$$
\begin{aligned}
x'(t) &= x(t) - 2y(t) \qquad x(0) = 1 \\
y'(t) &= -x(t) + y(t) \qquad y(0) = 1
\end{aligned}
$$

Thank You